§1. Differential Calculus on Complex Manifolds:

Definition: Let M be a Hausdorff second countable paracompact topological space. A differentiable (resp., holomorphic) atlas of dimension n is an indexed family of pairs $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$, where

(i) each $U_\alpha \subseteq X$ is open subset of X with $\bigcup_{\alpha \in \Lambda} U_\alpha = X$,

(ii) φ_α is a homeomorphism from U_α onto an open disk in $\mathbb{R}^n$ (resp., $\mathbb{C}^n$), and

(iii) for each $\alpha, \beta \in \Lambda$, the transition map

$$\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a differentiable (resp., holomorphic) map.

Two such differentiable (resp., holomorphic) atlases $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ and $\{(V_\beta, \psi_\beta)\}_\beta$ are said to be equivalent if for any α, β, the map

$$\varphi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(U_\alpha \cap V_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap V_\beta)$$

is differentiable (resp., holomorphic).

Definition: A differentiable (resp., holomorphic) manifold of dimension n is a Hausdorff second countable paracompact topological space M together with an equivalence class of differentiable (resp., holomorphic) atlas of dimension n.

Let X be a complex manifold / differentiable manifold.

<u>Definition</u>: A holomorphic (resp., complex/real C^∞) vector bundle of rank r on X is a complex manifold (resp., a differentiable manifold) E together with a surjective holomorphic map (resp., differentiable map)

$$p : E \longrightarrow X$$

whose fibers $E_x := p^{-1}(x)$ over each point $x \in X$ has a structure of a r dimensional complex (resp., real) vector space such that there is an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of X and holomorphic isomorphism (resp., C^∞ isomorphism)

$$\varphi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{C}^r \quad (\text{resp., } U_\alpha \times \mathbb{R}^r)$$

such that the following diagram commutes

$$\begin{tikzcd}
p^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha", "\cong"'] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowright"] & U_\alpha \times \mathbb{C}^r \ (\text{resp., } U_\alpha \times \mathbb{R}^r) \arrow[dl, "pr_1"] \\
& U_\alpha &
\end{tikzcd}$$

and the restrictions of φ_α to each fibers $p^{-1}(x)$ are <u>$\mathbb{C}$-linear</u>, (resp., $\mathbb{R}$-linear)

i.e.,
$$\varphi_\alpha\big|_{p^{-1}(x)} : E_x \longrightarrow \{x\} \times \mathbb{C}^r \simeq \mathbb{C}^r \quad (\text{resp., } \mathbb{R}^r)$$

is a $\mathbb{C}$-linear map (resp., $\mathbb{R}$-linear), for all $x \in U_\alpha$, and for all $\alpha \in \Lambda$.

<u>Definition</u>: Let $p_E : E \longrightarrow X$ and $p_F : F \longrightarrow X$ be two holomorphic (resp., complex C^∞, real C^∞) vector bundles on X. A vector bundle homomorphism from E

into F is a holomorphic (resp., complex C^∞, real C^∞) map $f : E \longrightarrow F$ such that $p_F \circ f = p_E$ and for each $x \in X$, the induced map of fibers

$$f_x : E_x \longrightarrow F_x$$

is $\mathbb{C}$-linear (resp., $\mathbb{C}$-linear, $\mathbb{R}$-linear).

Two vector bundles E and F are said to be isomorphic if there are two vector bundle isomorphisms $f: E \to F$ and $g : F \to E$ such that $g \circ f = Id_E$ and $f \circ g = Id_F$.

A vector bundle E on X is called trivial if it is isomorphic to $X \times \mathbb{C}^n \xrightarrow{pr_1} X$, for some integer $n \geq 1$.

Remark 1: If E is a real vector bundle of rank r on a differentiable manifold M, then $E \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector bundle of rank r on M, known as the complexification of E.

If we consider $E \otimes_{\mathbb{R}} \mathbb{C}$ as a real vector bundle of rank $2r$ on M, then $(E \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}} \cong_{\mathbb{R}} E \oplus E$.

Remark-2: If E is a holomorphic vector bundle of rank r on a complex manifold X, then forgetting its holomorphic structure, we may consider E as a complex C^∞ vector bundle of rank r on X.

However given a complex C^∞ vector bundle E of rank r on a complex manifold X, one can ask for criterion to induce a structure of a holomorphic vector bundle on E. We may come back to this question later

Let V be a real vector space. An almost complex structure on V is a $\mathbb{R}$-linear map $I : V \to V$ such that $I \circ I = -\mathrm{id}_V$.

e.g. if V is a $\mathbb{C}$-vector space, then

$$I : V \longrightarrow V$$
$$v \longmapsto iv \ , \quad \text{where } i = \sqrt{-1},$$

is an almost complex structure on V.

If a real vector space V admits an almost complex structure $I : V \to V$, then we can define a $\mathbb{C}$-linear structure on V by setting

$$(a+ib)\cdot v := a\cdot v + b\cdot I(v) \ , \ \forall\, a+ib \in \mathbb{C} \ \& \ v \in V.$$

Proposition: Let $U \subseteq \mathbb{C}^n$ be an open subset of $\mathbb{C}^n$. Let TU be the real tangent bundle on U considered as a real manifold of dimension $2n$. Then there is a natural vector bundle homomorphism

$$I : TU \longrightarrow TU$$

such that $I^2 = -\mathrm{Id}_{TU}$. In particular, I induces a natural complex structure on the tangent spaces of U.

Proof:- Let $\{z_j := x_j + i y_j\}_{j=1}^{n}$ be the standard holomorphic coordinates on an open nbd of $x_0 \in U \subseteq \mathbb{C}^n$. Then

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\}$$

forms a basis for real tangent spaces $T_x U$, $\forall x \in V_x \subseteq U$.

Sending $\dfrac{\partial}{\partial x_j} \longmapsto \dfrac{\partial}{\partial y_j}$ and $\dfrac{\partial}{\partial y_j} \longmapsto -\dfrac{\partial}{\partial x_j}$

we get a vector bundle homomorphism $I : TU \longrightarrow TU$ which satisfies $I^2 = -\mathrm{Id}_{TU}$.

Let M be a real differentiable manifold of dimension 2n.
Let TM be the real C^∞ tangent bundle on M.

<u>Definition</u>: An almost complex structure on M is a vector bundle homomorphism

$$J : TM \longrightarrow TM$$

such that $J^2 = -Id_{TM}$.

An almost complex manifold is a pair $X=(M,J)$ where M is a differentiable manifold together with an almost complex structure on it.

<u>Proposition</u>: Any complex manifold X admits a natural almost complex structure on its C^∞ tangent bundle on the underlying real smooth manifold.

<u>Proof</u>: Choose a holomorphic atlas

$$\left\{ \varphi_\alpha : U_\alpha \xrightarrow{\sim} U'_\alpha \underset{\text{open}}{\subseteq} \mathbb{C}^n \right\}_{\alpha \in \Lambda} \quad \text{with} \quad \bigcup_{\alpha \in \Lambda} U_\alpha = X$$

Then the almost complex structures on U'_α induces an almost complex structure on $\{TU_\alpha : \alpha \in \Lambda\}$, which glues together appropriately to give a vector bundle homomorphism $I : TM \to TM$ with $I^2 = -id_{TM}$, where TM is the real C^∞ tangent bundle on the underlying real manifold M of the complex manifold X.

Let $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle of M. Then the almost complex structure extends to a $\mathbb{C}$-linear map $I_{\mathbb{C}} : T_{\mathbb{C}}M \longrightarrow T_{\mathbb{C}}M$ s.t. $I_{\mathbb{C}}^2 = -Id_{T_{\mathbb{C}}M}$.

Fiberwise this gives a direct sum decomposition of complexified tangent spaces

$$(T_{\mathbb{C}}M)_x := T_x^{1,0}M \oplus T_x^{0,1}M$$

where $T_x^{1,0}M := \ker\left(I_{\mathbb{C},x} - i\, Id_{T_{\mathbb{C}}M}\right)$

and $T_x^{0,1}M := \ker\left(I_{\mathbb{C},x} + i\, Id_{T_{\mathbb{C}}M}\right)$

This direct sum decompositions glues appropriately to give rise to a direct sum decomposition of complex C^∞ vector bundles $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$.

Where $T^{1,0}M$ = Holomorphic part of $T_{\mathbb{C}}M$

and $T^{0,1}M$ = anti-holomorphic part of $T_{\mathbb{C}}M$.

Both $T^{1,0}M$ and $T^{0,1}M$ are complex vector bundles of rank n.

Let $X = (M, J)$ be an almost complex manifold.

Define: $\Lambda_{\mathbb{C}}^k X := \Lambda^k (T_{\mathbb{C}}M)^*$, $\forall\, k \geq 0$

and $\Lambda^{p,q} X := \Lambda^p (T^{1,0}M)^* \otimes_{\mathbb{C}} \Lambda^q (T^{0,1}M)^*$, $\forall\, p,q \geq 0$.

These are complex C^∞ vector bundles on X.

The sheaves of C^∞ sections of $\Lambda_{\mathbb{C}}^k X$ and $\Lambda^{p,q} X$ are denoted by $\mathcal{A}_X^k$ and $\mathcal{A}_X^{p,q}$, respectively.

Let $A^k(X) := \Gamma_{C^\infty}(X, \mathcal{A}^k_X)$ and $A^{p,q}(X) := \Gamma_{C^\infty}(X, \mathcal{A}^{p,q}_X)$.

Elements of $A^k(X)$ (resp., $A^{p,q}(X)$) are called differential k-forms (resp., (p,q)-forms) on X.

Proposition: There is a natural direct sum decomposition of complex C^∞ vector bundles

$$\Lambda^k_{\mathbb{C}} X := \Lambda^k T^*_{\mathbb{C}} X = \bigoplus_{p+q=k} \Lambda^{p,q} X$$

and

$$\mathcal{A}^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_X .$$

Moreover, $\overline{\Lambda^{p,q} X} = \Lambda^{q,p} X$ and $\overline{\mathcal{A}^{p,q}_X} = \mathcal{A}^{q,p}_X$.

Definition: Let $X = (M, J)$ be an almost complex manifold. Let $d : \mathcal{A}^k_{X,\mathbb{C}} \longrightarrow \mathcal{A}^{k+1}_{X,\mathbb{C}}$ be the $\mathbb{C}$-linear extension of the exterior differential. Then we define

$$\partial : \mathcal{A}^{p,q}_X \xrightarrow{d} \mathcal{A}^{p+q+1}_X = \bigoplus_{i+j=p+q+1} \mathcal{A}^{i,j}_X \xrightarrow{\pi_{p+1,q}} \mathcal{A}^{p+1,q}_X$$

and

$$\bar\partial : \mathcal{A}^{p,q}_X \xrightarrow{d} \mathcal{A}^{p+q+1}_X = \bigoplus_{i+j=p+q+1} \mathcal{A}^{i,j}_X \xrightarrow{\pi_{p,q+1}} \mathcal{A}^{p,q+1}_X$$

Then using the Leibniz rule for the exterior differential d, one can check that

$$\left.\begin{array}{ll} \text{(i)} & \partial(\alpha\wedge\beta) = \partial\alpha\wedge\beta + (-1)^{p+q}\alpha\wedge\partial\beta \\ \text{and (ii)} & \bar\partial(\alpha\wedge\beta) = \bar\partial\alpha\wedge\beta + (-1)^{p+q}\alpha\wedge\bar\partial\beta \end{array}\right\} \forall\, \alpha, \beta \in \mathcal{A}^{p,q}_X .$$

Remark: An almost complex manifold need not admit any complex manifold structure at all!

Proposition: Let $X = (M, I)$ be an almost complex manifold. Then the following are equivalent:

(i) $d\alpha = \partial\alpha + \bar{\partial}\alpha$, $\forall\, \alpha \in \mathcal{A}^k(X)$ & $k \geq 0$.

(ii) the following composite map

$$A^{1,0}(X) \xrightarrow{d} A^2(X) = A^{2,0}(X) \oplus A^{1,1}(X) \oplus A^{0,2}(X) \xrightarrow{\pi^{0,2}} A^{0,2}(X)$$

vanishes identically.

Definition: An almost complex structure $I: TM \to TM$ on a differentiable manifold M is said to be integrable if $\forall\, p \geq 0$ we can write $d\alpha = \partial\alpha + \bar{\partial}\alpha$, $\forall\, \alpha \in A^p(X)$.

Proposition: An almost complex structure $J: TM \to TM$ on M is integrable iff the antiholomorphic part of the complexified tangent bundle of M is preserved under the Lie bracket operation of vector fields; i.e.,

iff $$[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M.$$

Theorem (Newlander-Nirenberg): Any integrable almost complex structure on M is induced by a complex structure on M.

Proposition: Let $f : X \longrightarrow Y$ be a holomorphic map of complex manifolds. Then for each $k \geq 0$, f induces a map

$$f^* : A^k(Y) \longrightarrow A^k(X)$$
$$\alpha \longmapsto f^*\alpha$$

which respects the direct sum decomposition:

$$f^* : A^{p,q}(Y) \longrightarrow A^{p,q}(X) \quad \forall\ p, q \text{ with } p+q=k$$

Let X be a complex manifold. Let $\Omega_X^1 = \mathcal{H}om(TX, \mathcal{O}_X)$ be the holomorphic cotangent bundle on X.

For an integer $p \geq 1$, define $\Omega_X^p := \Lambda^p \Omega_X^1$.

Then the vector space $H^0(X, \Omega_X^p)$ of all holomorphic p-forms on X coincides with the $\mathbb{C}$-linear subspace

$$\{\alpha \in A^{p,0}(X) \mid \bar{\partial}\alpha = 0\} \subseteq A^{p,0}(X) \subseteq A^p(X).$$

Remark: Let $f : X \longrightarrow Y$ be a holomorphic map of complex manifolds. Then there are natural $\mathcal{O}_X$-module homomorphisms

$$TX \longrightarrow f^*TY$$

and

$$f^*\Omega_Y^1 \longrightarrow \Omega_X^1.$$

This induces a $\mathbb{C}$-linear map

$$H^0(Y, \Omega_Y^p) \longrightarrow H^0(X, \Omega_X^p), \quad \forall\ p \geq 0.$$

Definition: A holomorphic map $f : X \longrightarrow Y$ of complex manifolds is said to be smooth at $x \in X$ if the induced map of tangent spaces $T_x X \longrightarrow (f^*TY)_x = T_{f(x)} Y$ is surjective.

Remark: If f is smooth at every point of $f^{-1}(y)$, then $f^{-1}(y)$ is a smooth complex submanifold of X.

Definition: Let $X=(M,I)$ be a complex manifold. Then the $(p,q)^{th}$ Dolbeault cohomology of X is the complex vector space

$$H^{p,q}(X) := H^q(A^{p,\bullet}(X), \bar\partial) = \frac{\ker(\bar\partial : A^{p,q}(X) \to A^{p,q+1}(X))}{\operatorname{Im}(\bar\partial : A^{p,q-1}(X) \to A^{p,q}(X))}.$$

Remark: Since $H^0(X, \Omega_X^p) = \ker(\bar\partial : A^{p,0}(X) \to A^{p,1}(X))$, and the sheaves $\mathcal{A}_X^{p,q}$ are acyclic, using $\bar\partial$-Poincaré lemma one can conclude that the Dolbault cohomology of X computes the cohomologies of the sheaf Ω_X^p of holomorphic differential forms on X.

In other words, $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$, for all $p,q \geq 0$.

§ Vector Bundle Valued Differential Forms on X:

Let E be a holomorosphic vector bundle on a complex manifold X. Recall that,

$\mathcal{A}_X^0 := \Lambda^0(T_{\mathbb{C}}^* X) = \mathcal{C}_X^\infty$ is the sheaf of $\mathcal{C}^\infty$ functions on X.

For any integer $p,q \geq 0$, $\mathcal{A}_X^{p,q}$ is a sheaf of $\mathcal{A}_X^0$-modues on X. The sheaf of sections of E naturally a sheaf of $\mathcal{A}_X^0$-modules on X. Therefore, we can define a sheaf $\mathcal{A}^{p,q}(E)$ which associates an open subset $U \subseteq X$ the set

$$\mathcal{A}^{p,q}(U,E) := \Gamma(U, \mathcal{A}_X^{p,q} \otimes_{\mathcal{A}_X^0} E).$$

This is called the sheaf of differential (p,q)-forms on X with values in E.

Lemma: If E is a holomorphic vector bundle on a complex manifold X, then for each integer $p \geq 0$, there is a natural $\mathbb{C}$-linear operator

$$\bar{\partial}_E : \mathcal{A}^{p,q}(E) \longrightarrow \mathcal{A}^{p,q+1}(E)$$

satisfying the Leibniz rule

$$\bar{\partial}_E (f.\alpha) = \bar{\partial} f \wedge \alpha + f \bar{\partial}_E(\alpha)$$

for all $f \in \mathcal{A}_X^0$ and $\alpha \in \mathcal{A}^{p,q}(E)$,

and satisfies $\bar{\partial}_E \circ \bar{\partial}_E = 0$.

Proof: Fix a local holomorphic trivializing frame $s = (s_1, \dots, s_r)$ of E and write a section $\alpha \in \mathcal{A}^{p,q}(E)$ as $\alpha = \sum_{i=1}^{r} \alpha_i \otimes s_i$ with $\alpha_i \in \mathcal{A}_X^{p,q}$.

Then define $\bar{\partial}_E(\alpha) := \sum_{i=1}^{r} \bar{\partial}(\alpha_i) \otimes s_i$.

If $s' := (s_1', \dots, s_r')$ is another holomorphic trivializing frame field for E, then we obtain another operator $\bar{\partial}_E'$. We claim that $\bar{\partial}_E = \bar{\partial}_E'$. Indeed, if $(\varphi_{ij})_{1 \leq i,j \leq r}$ is the holomorphic transition matrix, i.e.,

$$s_i = \sum_{j=1}^{r} \varphi_{ij} \cdot s_j' \quad \forall\, i = 1, \dots, r,$$

then $\bar{\partial}_E \alpha = \sum_{i=1}^{r} \bar{\partial} \alpha_i \otimes s_i = \sum_{i=1}^{r} \bar{\partial} \alpha_i \otimes \sum_{j=1}^{r} \varphi_{ij}\, s_j'$.

Then $\bar{\partial}_E' \alpha = \bar{\partial}_E' \left(\sum_{i=1}^{r} \alpha_i \otimes s_i \right) = \bar{\partial}_E' \left(\sum_{i=1}^{r} \alpha_i \otimes \sum_{j=1}^{r} \varphi_{ij} \cdot s_j' \right)$

$$= \sum_{i,j=1}^{r} \bar{\partial}(\alpha_i \otimes \varphi_{ij}) \otimes s_j' = \sum_{i,j=1}^{r} \bar{\partial}\alpha_i \otimes \varphi_{ij} \otimes s_j' \quad (\because \bar{\partial}\varphi_{ij} = 0)$$

$$= \bar{\partial}_E \alpha.$$

(Proved)

Definition: The Dolbeault cohomology of a holomorphic vector bundle E over X is

$$H^{p,q}(X,E) := H^q(\mathcal{A}_X^{p,\bullet}(E), \bar\partial_E) = \frac{\ker(\bar\partial_E : \mathcal{A}_X^{p,q}(E) \to \mathcal{A}_X^{p,q+1}(E))}{\operatorname{Im}(\bar\partial_E : \mathcal{A}_X^{p,q-1}(E) \to \mathcal{A}_X^{p,q}(E))}.$$

Corollary: There is a natural isomorphism of $\mathbb{C}$-vector spaces

$$H^{p,q}(X,E) \cong H^q(X, E \otimes \Omega_X^p), \quad \forall\, p,q \geq 0.$$

Proof:- The complex of sheaves

$$\mathcal{A}_X^{p,0}(E) \xrightarrow{\bar\partial_E} \mathcal{A}_X^{p,1}(E) \xrightarrow{\bar\partial_E} \mathcal{A}_X^{p,2}(E) \to \cdots$$

is a resolution of $E \otimes \Omega_X^p$, and the sheaves $\mathcal{A}_X^{p,q}(E)$ are acyclic, $\forall\, p,q \geq 0$. Hence the result follows.

Remark: The following theorem may be considered as a linear version of the famous Newlander-Nirenberg theorem.

Theorem: Let E be a C^∞ complex vector bundle on a complex manifold X. Then E admits a holomorphic structure if and only if $\exists$ a $\mathbb{C}$-linear operator

$$\bar\partial_E : \mathcal{A}_X^0(E) \longrightarrow \mathcal{A}_X^{0,1}(E)$$

satisfying the Leibniz rule and with its natural $\mathbb{C}$-linear extension $\bar\partial_E : \mathcal{A}_X^{0,1}(E) \to \mathcal{A}_X^{0,2}(E)$ satisfying Leibniz rule, we have $\bar\partial_E \circ \bar\partial_E = 0$.