Kodaira's Embedding Theorem

1. What is Kodaira's embedding theorem?
2. Preliminaries from complex geometry:
   i. Connection on vector bundles.
   ii. Hermitian vector bundle and Chern connection.
   iii. Kahler manifolds.
   iv. Blow-up of a complex manifold.
   v. Positivity of a line bundle.
3. Proof of Kodaira's embedding theorem.

Introduction:

A complex manifold is said to be projective if it is a closed submanifold of some complex projective space $\mathbb{CP}^n$ or $\mathbb{P}^n$

Since any projective space is compact, any projective manifold is automatically compact. In general, geomety of a projective manifold is much easier to handle than arbitrary manifolds.

A complex projective space admits a natural Hermitian metric, called the Fubini-Study metric (to be defined later), which has a nice property: namely, it is a Kähler metric.
This makes a complex projective space a Kähler manifold.

Q. Are complex projective manifolds Kähler?

Given a complex submanifold Y of a Kähler manifold X, restricting its Kähler metric we get a Kähler metric on Y. This makes Y a Kahler manifold (To be discussed later). In particular, any complex projective manifold is Kähler.

However, there are examples of Kähler manifolds which are not projective.

Q. Is being a Kähler manifold a topological property?

For a compact complex surface, it is. But not in general.

Fact: A compact complex surface is Kähler if and only if its first betti number $b_1(X)$ is even.

However, this is not the case from dimension 3 onwards.

Hironaka constructed a family of examples of a compact complex non-Kähler manifolds of dimension 3 which are diffeomorphic to complex projective manifolds.

However, one can ask for criterion for a compact Kähler manifold to be a complex projective manifold.

This was answerd by Kodaira in his famous embedding theorem.

Kodaira's Embedding Theorem:

> A compact Kähler manifold X is projective if and only if X admits a positive line bundle.

Before going into the proof of this theorem, we need to understand Kähler manifolds, positive line bundles etc. So let's review some technologies from complex geometry.

Let E be a complex C^∞ vector bundle on a complex manifold X. Let $\mathcal{A}^0_X = \mathcal{C}^\infty_X$ be the sheaf of C^∞ functions on X. For $p \geq 1$, let $\mathcal{A}^p_X$ be the sheaf of differential p-forms on X. Define $\mathcal{A}^p_X(E) = \mathcal{A}^p_X \otimes E$, the sheaf of differential p-forms on X with values in E. For $p=0$, $\mathcal{A}^0_X(E) =$ sheaf of C^∞-sections of E.

Definition: A C^∞ connection on a complex vector bundle E over X is a $\mathbb{C}$-linear map

$$D : \mathcal{A}^0_X(E) \longrightarrow \mathcal{A}^1_X(E) = E \otimes \mathcal{A}^1_X$$

satisfying the Leibniz rule:

$$D(f \cdot s) = s \otimes df + f \cdot D(s),$$ for all locally defined sections $f \in \mathcal{A}_X^0$ and $s \in \mathcal{A}_X(E)$.

Let $E|_U \cong U \times \mathbb{C}^r$. Choose $s = (s_1, \ldots, s_r)$ with $s_i \in \Gamma(E|_U)$, such that $(s_1(x), \ldots, s_r(x))$ is an ordered basis for E_x, $\forall\, x \in U$.

Then $D(s_i) = \sum_{j=1}^{r} \omega_{ij} s_j$, for some $\omega_{ij} \in \Gamma(U, \mathcal{A}^1_X)$.

If $s = \sum_{i=1}^{n} f_i s_i$, for some $f_i \in \Gamma(U, \mathcal{A}^0_X)$, then we have

$$D(s) = \sum_{i=1}^{n} s_i \otimes df_i + f_i D(s_i) = \sum_{i=1}^{n} \Big(s_i \otimes df_i + f_i \sum_{j=1}^{n} \omega_{ij} s_j \Big)$$

$$D(s) = \sum_{i=1}^{n} s_i \otimes \Big(df_i + \sum_{j=1}^{n} f_j \omega_{ji} \Big)$$

$$s = \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix}$$

$$\therefore \boxed{Ds = ds + \omega s}$$

where $\omega = (\omega_{ij})_{1 \leq i,j \leq n}$ is a matrix of 1-forms on X.

This matrix is called the connection matrix.

Given a C^∞ connection $D : A^0(E) \to A^1(E)$ on E, extend it to a $\mathbb{C}$-linear operator

$$D : A^1(E) \longrightarrow A^2(E)$$

by setting $\boxed{D(s \otimes \alpha) = Ds \wedge \alpha + s \otimes d\alpha}$ for all $s \in \Gamma(E)$ and $\alpha \in \Gamma(A^1_X)$.

Then $D \circ D : A^0(E) \xrightarrow{D} A^1(E) \xrightarrow{D} A^2(E)$ is A^0_X-linear, and hence we can consider it as a section

$$R_D = D \circ D \in \Gamma_{C^\infty}(X, \mathrm{End}(E) \otimes A^2_X),$$

2-form on X with values in $\mathrm{End}(E)$. This is called the curvature of D. If $R_D = D \circ D = 0$, we call D a flat connection, and in this case (E, D) is called a flat vector bundle.

Definition: A hermitian metric on E is given by a C^∞ field of hermitian structure on the fibers of E. More precisely, for each $x \in X$ the fiber E_x admits a hermitian metric

$$h_x : E_x \times E_x \longrightarrow \mathbb{C}$$

(where (i) h_x is $\mathbb{C}$-linear in the first variable,
(ii) $h_x(v, v') = \overline{h_x(v', v)}$, $\forall v, v' \in E_x$,
(iii) $h_x(v, v) > 0$ for $v \neq 0$ in E_x)

such that for any C^∞ sections $\underline{\xi}, \underline{\eta} \in \Gamma(U, E)$, the map

$$h(\xi, \eta) : U \longrightarrow \mathbb{C}$$
$$x \longmapsto h_x(\xi_x, \eta_x)$$

is a C^∞ function on $U \underset{\text{open}}{\subseteq} X$.

The pair (E, h) is called a Hermitian vector bundle.

Definition: A C^∞ connection D on a Hermitian vector bundle (E, h) is called a h-connection if

$$h(D\xi, \eta) + h(\xi, D\eta) = \underline{d}h(\xi, \eta), \quad \forall \xi, \eta \in A_X^0(E).$$

where $d : A_X^0 \to A_X^1$ is the exterior differential operator.

Choose a local frame field $s_1, \dots, s_r \in \Gamma(U, E|_U)$.
Let $\underline{\omega} = (\omega_{ij})_{1 \le i,j \le n}$ be the matrix of the h-connection D_h on E. Then setting $h_{ij} = h(s_i, s_j)$ $\forall\, 1 \le i,j \le n$, we have

$$
\begin{aligned}
dh_{ij} &= h(Ds_i, s_j) + h(s_i, Ds_j) \\
&= \sum_{k=1}^{r} \omega_{ik} h_{kj} + \sum_{l=1}^{r} \overline{\omega_{jl}}\, h_{il}
\end{aligned}
$$

In matrix notation, $\boxed{dH = \omega^t H + H\bar{\omega}}$, where $H = (h_{ij})$.

$\Rightarrow 0 = d^2 H = d\omega^t \otimes H + H \otimes d\bar{\omega} \Rightarrow \boxed{\Omega^t H + H\bar{\Omega} = 0}$

where $\Omega = d\omega$ is the matrix of the curvature form $R_D = D \circ D$.

Proposition: Given a holomorphic vector bundle E on a complex manifold X, and a hermitian metric h on E, there is a unique h-connection $D_h : A^0(E) \to A^1(E)$ on E such that $\pi_{0,1} \circ D_h = \bar{\partial}_E$,

$$
\begin{tikzcd}
A^0(E) \arrow[r, "D"] \arrow[drr, "\bar{\partial}_E"'] & A^1(E) = A^{1,0}(E) \oplus A^{0,1}(E) \arrow[dr, "\pi_{0,1}"] & \\
& & A^{0,1}(E)
\end{tikzcd}
$$

where $\bar{\partial}_E$ is the holomorphic structure on E.

This unique connection D_h on E is called the Hermitian connection or Chern connection on E.

Remark: The Chern connection D_h is not holomorphic. In fact, its curvature R_{D_h} coincides with the Atiyah class $\mathcal{A}_E \in H^1(X, End(E) \otimes \Omega^1_X)$.
Therefore, E admits a holomorphic connection if and only if the cohomology class $[R_{D_h}] \in H^1(X, End(E) \otimes \Omega^1_X)$ is trivial.

- Note that, given any Chern connection D_h, any other Chern connection $D_{h'}$ on E differs by a C^∞ function, and so $R_{D_h} - R_{D_{h'}} = \bar{\partial} f$, for some $f \in \mathcal{A}^0_X$. So their cohomology classes are same.

Let

$$\begin{tikzcd}
 & & A^{1,0}(E) \\
A^0(E) \arrow[urr, bend left, "D' = \pi_{1,0} \circ D"] \arrow[rr, "D"] \arrow[drr, bend right, "D''"'] & & A^1(E) = A^{1,0}(E) \oplus A^{0,1}(E) \arrow[u, "\pi_{1,0}"'] \arrow[d, "\pi_{0,1}"] \\
 & & A^{0,1}(E)
\end{tikzcd}$$

Then $D = D' + D''$, and hence

$$R_D = D \circ D = D'^2 + D'D'' + D''D' + D''^2$$

Since $D''_h = \bar{\partial}_E$, $D''^2_h = \bar{\partial}^2_E = 0$.

Since $\boxed{\Omega^t H + H \bar{\Omega} = 0}$, where Ω is the matrix of R_D, we conclude that $D'^2_h = 0$, and hence

$$R_D = D \circ D = D' \circ D'' + D'' \circ D' \in \mathcal{A}^{1,1}_X \otimes End(E)$$

is a $(1,1)$-form on X with values in $End(E)$.

$I : TM \to TM \ni I^2 = -1$

A hermitian structure on $X = (M\ I)$ is given by a hermitian metric h on the holomorphic tangent bundle TX.

(Remark: Note that TX is isomorphic to $T^{1,0}M$ as complex vector bundles via the homomorphism

$$TX \xrightarrow{\sim} T^{1,0}M$$
$$\xi \mapsto \frac{1}{2}\left(\xi - \sqrt{-1}.\, I(\xi)\right)$$

[Note: A holomorphic section ξ of TX is a C^∞ section of $TM \otimes \mathbb{C}$])

Let $(z_1, \dots, z_n)$ be a holomorphic local coordinate system on $U \subseteq X$, and let $z_j = x_j + \sqrt{-1}\, y_j \ \forall 1 \le j \le n$.

Let
$$\left.\begin{aligned} \frac{\partial}{\partial z_j} &:= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right) \\ \text{and} \quad \frac{\partial}{\partial \bar{z}_j} &:= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right) \end{aligned}\right\} \forall 1 \le j \le n.$$

Then $\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)$ is a holomorphic local frame of $TX = T^{1,0}M$ and $\left(\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}\right)$ is a local frame field for $T^{0,1}M = \overline{T^{1,0}M}$.

Let $(dz_j = dx_j + i dy_j)_{j=1}^{n}$ be the holomorphic frame field for the holomorphic cotangent bundle T^*X dual to $\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)$. i.e., $dz_j\left(\frac{\partial}{\partial z_i}\right) = \delta_{ij} \ \forall i, j$.

Similarly, let $\left(d\bar{z}_j = dx_j - i\,dy_j\right)_{j=1}^{n}$ be the local frame field for $(T^{0,1}M)^*$ dual to $\left(\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}\right)$.

Let h be a Hermitian structure on X.

Considering h as a C^∞ section of the C^∞ vector bundle

$$(TX)^* \otimes (\overline{TX})^* = (T^{1,0}M)^* \otimes (T^{0,1}M)^*,$$

we can write
$$h = \sum_{i,j=1}^{n} h_{i\bar{j}}\, dz_i \otimes d\bar{z}_j,$$

where $h_{i\bar{j}} = h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right), \ \forall\, 1 \le i, j \le n.$

Definition: A hermitian structure h on $X = (M, I)$ is said to be **Kähler** if the induced real $(1,1)$-form (known as the **fundamental form of h**)

$$\omega_h := h(I(-), -)$$

is d-closed, i.e., $\boxed{d\omega_h = 0.}$

In terms of local holomorphic coordinates, ω_h can be written as

$$\omega_h = \sqrt{-1} \sum_{i,j=1}^{n} h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$$

where $h_{i\bar{j}} = h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right), \ \forall\, 1 \le i, j \le n.$

Remark: $\overline{\omega_h} = \omega_h$, and hence ω_h is a real $(1,1)$-form.

Remark: A hermitian metric h on X is Kähler if and only if the associated Chern connection D_h on TX is torsion free in the sense that

$$\mathrm{tor}(D_h)(\xi,\eta) := D_h(\xi)(\eta) - D_h(\eta)(\xi) - [\xi,\eta] = 0$$

for all locally defined sections ξ and η of TX.

A Kähler manifold is a complex manifold $X=(M,I)$ admitting a Kähler metric.

Remark: A complex manifold always admits a hermitian metric, but need not admit any Kähler metric at all!

Since $\omega_h = h(I(-),-)$, and h is compatible with the almost complex structure I on X in the sense that $h(I(-),I(-)) = h(-,-)$, we can recover the metric h from ω_h and I.

Indeed, $h(\xi,\eta) = h(I\xi, I\eta) = \omega_h(\xi, I\eta), \forall \xi,\eta$

$$\Rightarrow h(-,-) = \omega_h(-,I(-)).$$

Then we have the following:

Lemma: Let ω be a d-closed real $(1,1)$-form on a complex manifold X. If ω is positive definite (i.e., locally ω is of the form

$$\omega = \sqrt{-1} \sum h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$$

with $H = (h_{i\bar{j}})_{1 \le i,j \le n}$ a positive definite Hermitian matrix), then $\exists$ a Kähler metric h on X such that $\omega = \omega_h$, ω_h is the fundamental form of h.

Corollary: The set of all d-closed positive definite real $(1,1)$-forms in $A^{1,1}(X)$ are all Kähler forms on X.

Corollary: The set of all Kähler forms on X forms an open convex cone in the linear space

$$\{\omega \in A^{1,1}(X) \cap A^2(X,\mathbb{R}) \mid d\omega = 0\}$$

Proof: The positive definitness of a C^∞ family of hermitian matrices being open property, openness follows.
Since for any two Kähler forms ω_1, ω_2 we have $\omega_1 + \omega_2$ and $\lambda\omega_1$ is Kähler $\forall\, \lambda \in \mathbb{R}_{>0}$, the result follows.

Examples:

1. Fubini-Study metric on $\mathbb{P}^n_{\mathbb{C}}$ is a canonical Kähler metric on $\mathbb{P}^n_{\mathbb{C}}$. This is defined as follow:

Let $\mathbb{P}^n_{\mathbb{C}} = \bigcup_{i=0}^{n} U_i$ be the standard open covering where $\varphi_i : U_i \xrightarrow{\sim} \mathbb{C}^n$ is the homeomorphism

$$(z_0 : z_1 : \cdots : z_n) \mapsto \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i} \right) \in \mathbb{C}^n$$

Define $\omega_i := \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\left(\sum_{j=0}^{n} \left| \frac{z_j}{z_i} \right|^2 \right) \in A^{1,1}(U_i)$

Since $d = \partial + \bar{\partial}$ and $\bar{\partial}^2 = 0 = \partial^2$, we have $\partial\bar{\partial} + \bar{\partial}\partial = 0$.
Therefore, $\overline{\partial\bar{\partial}} = \bar{\partial}\partial = -\partial\bar{\partial}$, and hence $\bar{\omega}_i = \omega_i$, $\forall i$.
So ω_i is a real $(1,1)$-form on U_i, $\forall i$.
To show that $\{\omega_i \in A^{1,1}(U_i)\}_{i=0}^{n}$ glues together to give a global real $(1,1)$-form on $X = \mathbb{P}^n_{\mathbb{C}}$, note that

$$\log\left(\sum_{k=0}^{n} \left| \frac{z_k}{z_i} \right|^2 \right) = \log\left(\left| \frac{z_j}{z_i} \right|^2 \sum_{k=0}^{n} \left| \frac{z_k}{z_j} \right|^2 \right)$$
$$= \log |z_j/z_i|^2 + \log\left(\sum_{k=0}^{n} \left| \frac{z_k}{z_j} \right|^2 \right).$$

So it suffices to show that $\partial\bar{\partial}\left(\log |z_j/z_i|^2 \right) = 0$ on $U_i \cap U_j$.
Since z_j/z_i is the j^{th}-coordinate function on U_i, this follows from $\partial\bar{\partial} \log |z|^2 = \partial\bar{\partial} \log (z\bar{z})$

$$= \partial\left(\frac{1}{z\bar{z}} \bar{\partial}(z\bar{z}) \right) = \partial\left(\frac{z\, d\bar{z}}{z\bar{z}} \right) = 0.$$

$$\left[\begin{array}{l}\bar\partial(z\bar z) = \bar\partial(z)\cdot\bar z + z\,\bar\partial(\bar z) = 0 + z = z \\ \Rightarrow \partial\left(\dfrac{\bar\partial(z\bar z)}{z\bar z}\right) = \partial\left(\dfrac{z}{z\bar z}\right) = \partial\left(\dfrac{1}{\bar z}\right) = 0\end{array}\right]$$

Thus $\{\omega_i \in A^{1,1}(U_i)\}_{i=0}^{n}$ glues together to define a global real $(1,1)$-form $\omega_{FS} \in A^{1,1}(\mathbb{P}^n_{\mathbb{C}})$.

Now it remains to show that ω_i is positive definite, for all i. Setting $z_k/z_i = v_k$, and permuting indices, we see that

$$\partial\bar\partial \log\left(\sum_{k=0}^{n}\left|\frac{z_k}{z_i}\right|^2\right) = \partial\bar\partial \log\left(1+\sum_{i=1}^{n}|v_i|^2\right)$$

$$= \frac{\sum dv_i\wedge d\bar v_i}{1+\sum|v_i|^2} - \frac{\left(\sum \bar v_i dv_i\right)\wedge\left(\sum v_i d\bar v_i\right)}{\left(1+\sum|v_i|^2\right)^2}$$

$$= \frac{1}{\left(1+\sum|v_i|^2\right)^2}\sum_{i,j=1}^{n} h_{ij}\, dv_i\wedge d\bar v_j$$

where $h_{ij} = \left(1+\sum|v_i|^2\right)\delta_{ij} - \bar v_i v_j$.

Then one can check that the matrix $(h_{ij})_{1\le i,j\le n}$ is positive definite.

Note that for the natural projection

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^* ,$$

$$\pi^* \omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\|z\|^2).$$

Over $\pi^{-1}(U_i) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} \mid z_i \neq 0\}$,

we have,

$$\pi^* \omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\left(\sum_{k=0}^{n} \left|\frac{z_k}{z_i}\right|^2\right)$$

$$= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \left(\log \|z\|^2 - \log(|z_i|^2)\right)$$

Since $\partial\bar{\partial} \log(|z_i|^2) = 0$ (as we have seen above), we get $\pi^* \omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\|z\|^2)$.

Lemma: Let $Y \subset X$ be a complex submanifold of a complex manifold X. Then the restriction $h|_Y$ of any Kähler metric h on X is again Kähler.

Proof: Let I be the almost complex structure on X induced by its complex structure. Then $I_Y = I|_Y$ coincides with the almost complex structure induced from the complex structure of Y. If h is a Hermitian metric on X, and $I|_Y = I_Y$ preserves TY, $h_Y := h|_Y$ remains Hermitian metric compatible with I_Y, i.e., $h_Y(I_Y(-), I_Y(-)) = h_Y(-,-)$.

Then $d_Y(\omega_{h_Y}) = d_Y h_Y(I_Y(-),(-)) = d_X h(I(-),(-))|_Y = 0$ and hence ω_{h_Y} is Kähler.

Corollary: Any projective manifold is Kähler.

Proof: Since projective manifolds are by definition closed submanifolds of a projective space, restricting Fubini-Study metric, we get the required Kähler metric.

Remark: There could be many Kähler metric on a submanifold $Y \subseteq X$, which do not comes as a restriction of some Kähler metric on X.