Towards Kodaira's Embedding Theorem

Lecture 2

Contents:

Recap. Let X be a complex manifold. Let M be the underlying real differentiable manifold of X. Then the complex structure on X induces an integrable almost complex structure $I : TM \to TM$ with $I^2 = -\mathrm{Id}_{TM}$.
Tensoring with $\mathbb{C}$, $I : TM \otimes \mathbb{C} \longrightarrow TM \otimes \mathbb{C}$ gives eigen space decomposition $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$, with $T^{1,0}M \cong TX$, the holomorphic tangent bundle of X.

Definition: A hermitian structure h on $X = (M, I)$ is said to be Kähler if the induced real $(1,1)$-form

$$\omega_h := h(I(-), -)$$

is d-closed, i.e., $d\omega_h = 0$.

The real $(1,1)$-form ω_h is known as the fundamental form of h, which is called a Kähler form if $d\omega_h = 0$.

Since a hermitian metric h on the holomorphic tangent bundle TX of X preserves/respects the induced almost complex structure I (i.e., $h(I\xi, I\eta) = h(\xi, \eta)$, for all locally defined sections ξ, η of TX, we can recover a hermitian metric h on TX from its fundamental form ω_h and the almost complex structure I by the formula:

$$h(-,-) = \omega_h(-, I(-)).$$

Last time we have discussed explicit construction of the Fubini-Study Kähler form ω_{FS} on the complex projective space $\mathbb{P}^n_{\mathbb{C}}$, which on the standard open covering $\mathbb{P}^n_{\mathbb{C}} = \bigcup_{i=0}^{n} U_i$ with $U_i \cong \mathbb{C}^n$,

$$(z_0 : z_1 : \dots : z_n) \longmapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i} \right) \in \mathbb{C}^n$$

is given by the formula

$$\omega_{FS}\Big|_{U_i} := \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\left(\sum_{j=0}^{n} \left| \frac{z_j}{z_i} \right|^2 \right) \in A^{1,1}(U_i), \ \forall i.$$

Now we show how ω_{FS} defined on $\mathbb{P}^n_{\mathbb{C}}$ induces a Kähler structure on $\mathbb{C}^{n+1} \setminus \{0\}$ via the natural projection map

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n_{\mathbb{C}} = \left(\mathbb{C}^{n+1} \setminus \{0\} \right) / \mathbb{C}^*$$

Over $\pi^{-1}(U_i) = \{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} \mid z_i \neq 0 \}$, we have,

$$\pi^* \omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{k=0}^{n} \left| \frac{z_k}{z_i} \right|^2 \right)$$

$$= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \left(\log \|z\|^2 - \log(|z_i|^2) \right)$$

Since $\partial\bar{\partial} \log(|z_i|^2) = 0$ (as we have seen earlier) we get $\pi^* \omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\|z\|^2)$.

Lemma: Let $Y \subset X$ be a complex submanifold of a complex manifold X. Then the restriction $h|_Y$ of any Kähler metric h on X is again Kähler.

Proof: Let I be the almost complex structure on X induced by its complex structure. Then $I_Y = I|_Y$ coincides with the almost complex structure induced from the complex structure of Y. If h is a Hermitian metric on X, and $I|_Y = I_Y$ preserves TY, $h_Y := h|_Y$ remains Hermitian metric compatible with I_Y, i.e., $h_Y(I_Y(-), I_Y(-)) = h_Y(-,-)$.

Then $d_Y(\omega_{h_Y}) = d_Y\, h_Y(I_Y(-),(-)) = d_X\, h(I(-),(-))|_Y = 0$ and hence ω_{h_Y} is Kähler.

Corollary: Any projective manifold is Kähler.

Proof: Since projective manifolds are by definition closed submanifolds of a projective space, restricting Fubini-Study metric, we get the required Kähler metric.

Remark: There could be many Kähler metrics on a submanifold $Y \subseteq X$, which do not come as a restriction of some Kähler metric on X.

Lemma: Let X be a compact Kähler manifold. Let E be a holomorphic vector bundle on X, and $\mathbb{P}(E)$ be the associated projective bundle over X. Then the compact complex manifold $\mathbb{P}(E)$ is Kähler.

Sketch of a proof:-

Let $\pi : \mathbb{P}(E) \longrightarrow X$ be the natural projection. Let $\mathcal{O}_{\mathbb{P}(E)}(-1)$ be the tautological line subbundle of π^*E over $\mathbb{P}(E)$ whose fiber over a point $l_x \in \mathbb{P}(E_x) \subset \mathbb{P}(E)$, $x \in X$, is the line $l_x \subset E_x$ itself. Define $\mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_{\mathbb{P}(E)}(-1)^\vee$.

Note that $\mathcal{O}_{\mathbb{P}(E)}(1)\big|_{\mathbb{P}(E_x)} \cong \mathcal{O}_{\mathbb{P}(E_x)}(1)$, where $\mathbb{P}(E_x) = \pi^{-1}(x)$, $\forall x \in X$.

Let h be a Hermitian metric on E. Then h induces a Hermitian metric π^*h on π^*E. Since $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is a line subbundle of π^*E, we can restrict π^*h to get a Hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(-1)$, which again induces a Hermitian metric, say $\tilde{h}$, on the dual line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Let $\omega_{\tilde{h}}$ be the fundamental form of $\tilde{h}$.
Since the restriction of $\tilde{h}$ over each fiber $\pi^{-1}(x) = \mathbb{P}(E_x)$ coincides with the Fubini-Study Kähler metric on $\mathbb{P}(E_x) \cong \mathbb{P}^n_{\mathbb{C}}$ induced from the Hermitian metric h_x on the fiber E_x of E, the fundamental form $\omega_{\tilde{h}}\big|_{\mathbb{P}(E_x)}$ remains positive on $\mathbb{P}(E_x) = \pi^{-1}(x)$, $\forall x$.

Since X is a compact Kähler manifold, we have a Kähler form ω_X on X.
Then X being compact, one can find a positive real number $\lambda > 0$ such that the real closed $(1,1)$-form

$$\omega := \omega_{\tilde{h}} + \lambda \pi^* \omega_X$$

is positive on $\mathbb{P}(E)$. This completes the proof.

$\square$

Blow-up of a complex manifold along a closed submanifold.

Let X be a complex manifold of dimension n. Let $Y \subseteq X$ be a closed complex submanifold of dimension $m < n$. Then **blow-up of X along Y** is a complex manifold $\tilde{X} := Bl_Y(X)$ together with a holomorphic map

$$\sigma : Bl_Y(X) \longrightarrow X$$

such that $\sigma|_{\tilde{X} \setminus \sigma^{-1}(Y)} : Bl_Y(X) \setminus \sigma^{-1}(Y) \xrightarrow{\cong} X \setminus Y$ is an isomorphism of complex manifolds and

$$\sigma|_{\sigma^{-1}(Y)} : \sigma^{-1}(Y) \longrightarrow Y$$

is the projective bundle $\mathbb{P}(N_{Y/X}) \longrightarrow Y$, where $N_{Y/X} = (\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$ is the normal bundle of $Y \underset{\text{closed}}{\hookrightarrow} X$.

Construction of blow-up $\sigma : Bl_Y(X) \longrightarrow X$.

Case-1 : Let $X = \mathbb{C}^{n+1}$, $Y = \{0\} \subseteq \mathbb{C}^{n+1}$; $n \geq 1$.

Let

$$\begin{tikzcd} Z := \{(z, l) \in \mathbb{C}^{n+1} \times \mathbb{P}^n_{\mathbb{C}} \mid z \in l\} \arrow[r, "pr_2"] \arrow[d, "pr_1"] & \mathbb{P}^n_{\mathbb{C}} \\ \mathbb{C}^n & \end{tikzcd}$$

Let $(x_0 : x_1 : \cdots : x_n)$ be the standard homogeneous local coordinate system on $\mathbb{P}^n_{\mathbb{C}}$. Let $(z_0, \ldots, z_n)$ be the holomorphic coordinates in $\mathbb{C}^{n+1}$. Then Z is defined by the relations $\{z_i x_j = z_j x_i \mid 0 \leq i, j \leq n\}$.

Clearly $pr_1^{-1}(z) = \begin{cases} \mathbb{P}^n & \text{if } z = 0 \text{ in } \mathbb{C}^{n+1} \\ \{*\} & \text{if } z \neq 0 \text{ in } \mathbb{C}^{n+1} \end{cases}$

Also $pr_2 : Z = \{(z,l) \in \mathbb{C}^{n+1} \times \mathbb{P}^n \mid z \in l\} \longrightarrow \mathbb{P}^n$ is a locally trivial family of lines in $\mathbb{C}^{n+1}$, and so is a line bundle on $\mathbb{P}^n_{\mathbb{C}}$; its sheaf of sections is denoted by $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-1)$.

Case - II : Let $X = \mathbb{C}^n$ & $Y = \mathbb{C}^m$ embeded inside $X = \mathbb{C}^n$ by the equations $z_{m+1} = \dots = z_n = 0$, where $(z_1, \dots, z_n)$ is the standard local coordinates on $\mathbb{C}^n$, and $1 \leq m < n$.

Consider the incidence variety

$$Z := \{(z, l) \in \mathbb{C}^n \times \mathbb{P}(\mathbb{C}^{n-m}) : z \in \langle Y, l \rangle\}$$

Where $\langle Y, l \rangle = \mathbb{C}$- linear space generated by Y & l.

Note that Z is defined by the set of equations:

$$\{ z_i x_j - z_j x_i = 0 \mid i, j = m+1, \dots, n \}$$

Let $pr_1 : Z \longrightarrow X = \mathbb{C}^n$ and $pr_2 : Z \longrightarrow \mathbb{P}^{n-m-1}_{\mathbb{C}}$ be the projection maps.

Clearly, $pr_2^{-1}(l) = \mathbb{C}^{m+1}$, $\forall\, l \in \mathbb{P}^{n-m-1}_{\mathbb{C}}$

and one can check that $pr_2 : Z \longrightarrow \mathbb{P}^{n-m-1}$ is a holomorphic vector bundle of rank $m+1$.

By construction,

$$pr_1^{-1}(z) = \begin{cases} \mathbb{P}(N_{Y/X}|_z) = \mathbb{P}_{\mathbb{C}}^{n-m-1} & \text{if } z \in Y \\ \{*\} & \text{if } z \in X \setminus Y. \end{cases}$$

<u>Case III</u>: Assume X, Y are general. Choose a holomorphic atlas: $X = \bigcup_{i \in I} U_i$, $\varphi_i : U_i \xrightarrow{\sim} \tilde{U}_i \underset{\text{open}}{\subseteq} \mathbb{C}^n$,

$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$

is holomorphic $\forall i, j \in I$,

such that $\varphi_i(U_i \cap Y) = \varphi_i(U_i) \cap \mathbb{C}^m$, $\forall i \in I$,

where $\mathbb{C}^m \hookrightarrow \mathbb{C}^n$ by $z_{m+1} = \cdots = z_n = 0$.

Let $p : Bl_{\mathbb{C}^m}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n$ be the blow-up of $\mathbb{C}^n$ along $\mathbb{C}^m$.

Consider

$$\begin{tikzcd} Z_i := p^{-1}(\varphi_i(U_i)) \arrow[r, hook] \arrow[d, "p_i := p|_{Z_i}"] & Bl_{\mathbb{C}^m}(\mathbb{C}^n) \arrow[d, "p"] \\ \varphi_i(U_i) \arrow[r, hook] & \mathbb{C}^n \end{tikzcd}$$

Then glue these $\{(Z_i, p_i)\}_{i \in I}$ to construct the required blow-up map

$$\sigma : Bl_Y(X) \longrightarrow X.$$

Gluing of local blow-ups to construct global one:

Let $U, V \subseteq \mathbb{C}^n$ be any two open subsets together with a biholomorphism $\varphi = (\varphi^1, \dots, \varphi^n) : U \xrightarrow{\sim} V$.

$$z \mapsto (\varphi^1(z), \dots, \varphi^n(z))$$

Define a biholomorphism $\tilde{\varphi} : p^{-1}(U) \xrightarrow{\sim} p^{-1}(V)$ inside $Bl_{\mathbb{C}^m}(\mathbb{C}^n)$ by sending $(z, x) \in p^{-1}(U) \longmapsto (\varphi(z), A_\phi(x)) \in Bl_{\mathbb{C}^m}(\mathbb{C}^n)$, where $A_\phi = (\psi^i_j)_{m+1 \le i,j \le n}$ is the $(n-m)\times(n-m)$ matrix of holomorphic functions ψ^i_j obtained from the power series expansions

$$\varphi^i(z) = \sum_{j=m+1}^{n} z_j \psi^i_j(z_1, \dots, z_n), \quad \forall\, i = m+1, \dots, n,$$

and x is the $(n-m)\times 1$ column matrix $(x_{m+1} \cdots x_n)^t$ so that $A_\phi x$ is a $(n-m)\times 1$ column matrix.

Then one can check that the image of $\tilde{\varphi} : p^{-1}(U) \to Bl_{\mathbb{C}^m}(\mathbb{C}^n)$ lands inside $p^{-1}(V) \subseteq Bl_{\mathbb{C}^m}(\mathbb{C}^n)$, and is a biholomorphism $\tilde{\varphi} : p^{-1}(U) \xrightarrow{\sim} p^{-1}(V)$.

Now apply this construction to the transition maps (which are also biholomorphisms)

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \xrightarrow{\sim} \varphi_i(U_i \cap U_j)$$

to get a biholomorphism $\tilde{\varphi}_{ij} : Z_i \cap Z_j \xrightarrow{\sim} Z_i \cap Z_j \quad \forall\, i, j \in I$

It remains to show that $\tilde{\varphi}_{ij}$ satisfies cocycle condition. Since the blowup map $p : Bl_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{C}^n$ is an iso over $\mathbb{C}^n \setminus \mathbb{C}^m$, we only need to check cocycle condition over $Y \hookrightarrow X$. But over Y, the matrices we obtain for each $\psi^i_j|_{\mathbb{C}^m}$ are by definition the matrices of transition functions for $N_{Y/X}$. Hence $\{ Z_i := p^{-1}(\varphi_i(U_i)) \xrightarrow{p_i} \varphi_i(U_i) \}_{i \in I}$ glues appropriately to give the required blow-up map $\sigma : Bl_Y(X) \longrightarrow X$. ◪

Remark: It follows that blow-up of a one dimensional complex manifold X along a finite subset of points of X is X itself.

Definition:

The hypersurface $\sigma^{-1}(Y) = \mathbb{P}(N_{Y/X}) \subset Bl_Y(X)$ is called the exceptional divisor of the blow-up morphism $\sigma : Bl_Y(X) \longrightarrow X$.

Remark:

If Y is a smooth divisor in X, its normal bundle $N_{Y/X}$ is a line bundle over Y, and hence the exceptional locus $\sigma^{-1}(Y) = \mathbb{P}(N_{Y/X}) \xrightarrow{\sim} Y$, which implies $\sigma : Bl_Y(X) \xrightarrow{\cong} X$ is an isomorphism.

Lemma: Let Y be a compact complex submanifold of a complex manifold X. Let $\sigma : \tilde{X} := Bl_Y(X) \longrightarrow X$ be the blow-up of X along Y. Then there is a holomorphic line bundle $\mathcal{L}$ on $\tilde{X}$ which is trivial outside the exceptional divisor $E := \sigma^{-1}(Y) = \mathbb{P}(N_{Y/X}) \subset \tilde{X}$, and $\mathcal{L}|_{\sigma^{-1}(Y)} \cong \mathcal{O}_{\mathbb{P}(N_{Y/X})}(1)$.

Sketch of a proof:-

Let D be a hypersurface in a complex manifold X; this means, D is locally defined by a holomorphic function, which is unique up to multiplication by an invertible function (i.e., a section of $\mathcal{O}_X^*$). Consider an open covering $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ of X such that $D \cap U_\alpha$ is defined by an equation $f_\alpha = 0$ in U_α, for some holomorphic function $f_\alpha \in \mathcal{O}_X(U_\alpha)$.

Of course we may take $U = X \setminus D$ as one of U_α and $f_U = 1$ on $U = X \setminus D$. Then for any other open set V from that covering of X, $g_{UV} := f_U / f_V$ is an invertible function on $U \cap V$.

Since they satisfies cocycle condition on triple intersections $U \cap V \cap W$, we can use these $\{g_{UV}\}$ to construct a line bundle on X whose transition functions are $\{g_{UV}\}$.

This line bundle is usually denoted by $\mathcal{O}_X(-D)$.
By its construction $\mathcal{O}_X(-D)\big|_{X\setminus D}$ is trivial.

Moreover, if $D \subset X$ is a smooth hypersurface in X, then $\mathcal{O}_X(-D)\big|_D \cong N^*_{Y/X}$, where $N^*_{Y/X} = \mathcal{I}_Y/\mathcal{I}_Y^2$ is the conormal bundle of $D \xhookrightarrow[\text{embedding}]{\text{closed}} X$.

This is because by differentiating the local defining equations for $D \hookrightarrow X$, we get the transition functions for the conormal bundle

$$N^*_{Y/X} = \mathcal{I}_Y/\mathcal{I}_Y^2 .$$

Now apply this construction to the smooth hypersurface $E := \sigma^{-1}(Y) \cong \mathbb{P}(N_{Y/X}) \subset Bl_Y(X) =: \tilde{X}$ to obtain $\mathcal{O}_{\tilde{X}}(-E)\big|_E \cong N^\vee_{E/\tilde{X}}$. Therefore, it is enough to show that the normal bundle $N_{E/\tilde{X}}$ to the exceptial divisor $E \cong \mathbb{P}(N_{Y/X}) \subset \tilde{X}$ is isomomorphic to the tautological line bundle $\mathcal{O}(-1)$ of the projective bundle $E = \mathbb{P}(N_{Y/X}) \xrightarrow[\sigma]{} Y$.

This can be checked from the description of

$\mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1) \hookrightarrow \sigma^* N_{Y/X}$ and local description of blow-up map $\tilde{X} \xrightarrow{\sigma} X$.

$\blacksquare$