Proof of Kodaira's Embedding Theorem

Lecture - 3 (Last talk)

Contents:

Recall that, last time we have discussed a construction of blow-up $\sigma : Bl_Y(X) \longrightarrow X$ of a complex manifold X along a closed submanifold Y of X. For notational simplicity, if we write $\tilde{X} = Bl_Y(X)$, then

$$\sigma|_{\tilde{X}\setminus\sigma^{-1}(Y)} : \tilde{X}\setminus\sigma^{-1}(Y) \xrightarrow{\sim} X\setminus Y$$

is an isomorphism of complex manifolds and $\sigma^{-1}(Y) \xrightarrow{\sigma} Y$ is isomorphic to the projective bundle $\mathbb{P}(N_{Y/X}) \longrightarrow Y$, where $N_{Y/X}$ is the normal bundle of the closed embedding $Y \hookrightarrow X$. The hypersurface $E := \sigma^{-1}(Y) \subset \tilde{X}$ is called the exceptional divisor of the blow-up $\sigma : \tilde{X} \to X$.

Then we have proved the following important result:

Lemma: With the above notations, there is a line bundle $\mathcal{L} := \mathcal{O}_{\tilde{X}}(-E)$ on $\tilde{X}$ which is trivial outside the exceptional divisor $E \subset \tilde{X}$, and over $E = \mathbb{P}(N_{Y/X})$, we have

$$\mathcal{O}_{\tilde{X}}(-E)|_E \cong \mathcal{O}_{\mathbb{P}(N_{Y/X})}(1).$$

This result is key to many important constructions we shall see later.

Recall that, a closed real $(1,1)$-form $\alpha \in A^{1,1}_X$ is called positive (or, semi-positive) if $-i\,\alpha(v,\bar{v}) > 0$ (resp., $-i\,\alpha(v,\bar{v}) \geq 0$), for all $0 \neq v \in T_x X = T^{1,0}_x X$.

Recall that the curvature Θ_h of the Chern connection D_h on a Hermitian vector bundle (E,h) on a complex manifold X is a purely imaginary d-closed $(1,1)$-form

$$\Theta_h = D'_h \circ D''_h + D''_h \circ D'_h \in A^{1,1}_X(\mathrm{End}(E)).$$

The associated fundamental form $\omega_h := -i\,\Theta_h$ is a real d-closed $(1,1)$-form on X with values in $\mathrm{End}(E)$.

<u>Definition</u> :- A holomorphic line bundle L on a complex manifold X is said to be positive if there is a Hermitian metric h on L such that the induced real $(1,1)$ closed form $i\,\Theta_h \in A^{1,1}_X$ is positive in the above sense, where Θ_h is the curvature of the Chern connection D_h on L.

In other words, $\Theta_h = -i \cdot i\,\Theta_h$ should satisfies $\Theta_h(v,\bar{v}) > 0 \;\; \forall\, 0 \neq v \in T_x X$.

§ Kähler structure under blow-up.

Lemma: If $Y \subseteq X$ is a compact complex submanifold of a Kähler manifold X, then the blowup $Bl_Y(X)$ of X along Y is again a Kähler manifold. Moreover, $Bl_Y(X)$ is compact if X is compact.

Sketch of a proof:

Since Y is compact, and the blow-up map $\sigma: Bl_Y(X) \longrightarrow X$ restricts to an isomorphism on $X \setminus Y$,

$$\sigma: \quad Bl_Y(X) \setminus \sigma^{-1}(Y) \xrightarrow{\cong} X \setminus Y,$$

and over Y, its the projective bundle

$$\mathbb{P}(N_{Y/X}) = \sigma^{-1}(Y) \xrightarrow{\sigma} Y$$

where $N_{Y/X}$ is the normal bundle of $Y \hookrightarrow X$ over Y, it follows that $Bl_Y(X)$ is compact iff X is compact.

Remark: The construction of a Kähler structure on $\tilde{X} = Bl_Y(X)$ is exactly similar to the construction for the case of projective bundle $\mathbb{P}(N_{Y/X}) \to Y$, we have discussed last time. Here we need to use the additional data: "$\exists$ a line bundle $\mathcal{O}_{\tilde{X}}(-E)$ on $\tilde{X}$ which is trivial outside $E = \sigma^{-1}(Y) \cong \mathbb{P}(N_{Y/X})$ and $\mathcal{O}_{\tilde{X}}(-E)|_E \cong \mathcal{O}_E(1)$." to get the required hermitian metric on whole $\tilde{X} = Bl_Y(X)$.

Details: Let ω_X be a Kähler form on X. Then $\sigma^*\omega_X$ is a real closed $(1,1)$-form on $\tilde{X} := Bl_Y(X)$, which is positive outside $E := \sigma^{-1}(Y) \subset Bl_Y(X)$. However, $\sigma^*\omega_X$ remains semi-positive along $\sigma^{-1}(Y)$.

The kernel of the pulled-back form $\sigma^*\omega_X$ over each point of $z \in \sigma^{-1}(Y)$ consists of the tangent spaces to the fibers of σ; i.e., $T_z\sigma^{-1}(\sigma(z))$.

Note that, if we have a real closed $(1,1)$-form λ on $\tilde{X} = Bl_Y(X)$, which is zero outside a compact neighbourhood of $\sigma^{-1}(Y) = E \subset Bl_Y(X)$ and strictly positive on the fibers $\sigma^{-1}(y)$, $\forall y \in Y$, then using compactness of $Y \subset X$, we can find a positive real number $c_0 \gg 0$ such that the real closed $(1,1)$-form $c_0\sigma^*\omega_X + \lambda$ is positive at each point of $Bl_Y(X) = \tilde{X}$. This gives us the required Kähler structure on $\tilde{X}$.

So it remains to construct such a λ on $\tilde{X}$. Recall that blow-up map is a projective bundle$/Y$ when restricted to the exceptional divisor $E = \sigma^{-1}(Y)$

$$\sigma^{-1}(Y) = \mathbb{P}(N_{Y/X}) \xrightarrow{\sigma} Y,$$

and there is a line bundle $\mathcal{L} = \mathcal{O}_{\tilde{X}}(-E)$ on $\tilde{X} = Bl_Y(X)$, which is trivial outside $E = \sigma^{-1}(Y)$, and over $E = \sigma^{-1}(Y)$ it is $\mathcal{O}_{\tilde{X}}(-E)|_E \cong \mathcal{O}_{\mathbb{P}(N_{Y/X})}(1)$.

Now recall that starting with a Hermitian metric on $N_{Y/X}$, we can construct a Hermitian metric $\tilde{h}$ on $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(1)$ such that the restriction of its associated fundamental form $\omega_{\tilde{h}}$ over each fiber $\sigma^{-1}(y) = \mathbb{P}(N_{Y/X}|_y)$ is a positive real closed (1,1)-form (Fubini-Study Kähler form on the projective space $\mathbb{P}(N_{Y/X}|_y)$).

Now using a C^∞ partition of unity, we can extend this Hermitian metric $\tilde{h}$ on $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(1)$ to a Hermitian metric, say $\hat{h}$ on the line bundle $\mathcal{L} = \mathcal{O}_{\tilde{X}}(-E)$, such that the associated Chern connection $D_{\hat{h}}$ become flat outside a relatively compact nbd. of E

[Note that, $\mathcal{O}_{\tilde{X}}(-E)|_{\tilde{X} \setminus E}$ is trivial.]

Therefore, this $\hat{h}$ gives our required real closed (1,1)-form λ on $\tilde{X}$, which is zero outside a relatively compact nbd. of $E = \sigma^{-1}(Y)$ and its restriction to each fiber $\sigma^{-1}(y)$, $y \in Y$, are positive. This completes the proof.] $\boxtimes$

Remark:- Converse of the above result is not true. There are examples of compact non-Kähler complex manifolds whose blow-up along a compact complex submanifold can be projective (and hence Kähler).

Lemma:- Let X be a complex manifold and $x \in X$. Let $\sigma : \tilde{X} = Bl_x(X) \longrightarrow X$ be the blow-up of X at x. Then $K_{\tilde{X}} \cong \sigma^* K_X \otimes \mathcal{O}_{\tilde{X}}((n-1)E)$, where $E = \sigma^{-1}(x) \subset \tilde{X}$ is the exceptional divisor and $n = \dim_{\mathbb{C}}(X) \geq 2$.

Sketch of a proof:-

Since $\sigma : \tilde{X} \setminus E \xrightarrow{\sim} X \setminus \{x\}$ is an isomorphism of complex manifolds, we have

$$K_{\tilde{X}} \cong \sigma^* K_X \otimes \mathcal{O}_{\tilde{X}}(E)^{\otimes a}, \qquad \text{-----}(1)$$

for some $a \in \mathbb{Z}$. The adjunction formula for the smooth hypersurface $j : E \hookrightarrow \tilde{X}$ gives

$$K_E \cong j^* K_{\tilde{X}} \otimes N_{E/\tilde{X}}, \qquad \text{----}(2)$$

where $N_{E/\tilde{X}}$ is the normal (line) bundle of $j : E \hookrightarrow \tilde{X}$.

Since $E \cong \mathbb{P}^{n-1}_{\mathbb{C}} \hookrightarrow \tilde{X}$ is a smooth hypersurface, we have

$N_{E/\tilde{X}} \cong \mathcal{O}_{\tilde{X}}(E)|_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$, by previous lemma.

Since $E \cong \mathbb{P}^{n-1}_{\mathbb{C}}$, $K_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-(n-1)-1) = \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(-n)$.

Putting all these together in eq$\underline{\underline{n}}$ (2), using eq$\underline{\underline{n}}$(1), we have,

$$\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(-n) = K_E = j^* \sigma^* K_X \otimes j^* \mathcal{O}_{\tilde{X}}(E)^{\otimes a} \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(-1).$$

Since the composition $E \overset{j}{\hookrightarrow} \tilde{X} \xrightarrow{\sigma} X$ is a constant map, we have

$$\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(-n) \cong \mathcal{O}_{\tilde{X}}(E)\Big|_E^{\otimes a} \otimes \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(-1)$$

$$\cong \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(-1)^{\otimes a} \otimes \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(-1)$$

$$\cong \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}\big(-(a+1)\big)$$

Since $\mathrm{Pic}(\mathbb{P}^{n-1}_{\mathbb{C}}) \cong \mathbb{Z}$, we have $n = a+1$.

Then from eq$^{\underline{n}}$ 1, we get

$$K_{\tilde{X}} \cong \sigma^* K_X \otimes \mathcal{O}_{\tilde{X}}((n-1)E). \qquad \text{(Proved)}$$

2.§ Linear system and embedding into a projective space:

Let X be a compact complex manifold. Let L be a holomorphic line bundle on X. X being compact, $H^0(X,L)$ is a finite dimensional $\mathbb{C}$-vector space. The associated projective space $|L| := \mathbb{P}(H^0(X,L)) = \left(H^0(X,L)\setminus\{0\}\right)/\mathbb{C}^*$ is called the complete linear system of L. A $\mathbb{C}$-linear subspace $V \subset H^0(X,L)$ defines a subset

$$\delta_V := \mathbb{P}(V) \subseteq |L|,$$

known as the linear system of $V \subset H^0(X,L)$.

Definition: A point $x \in X$ is said to be a base point of a linear system δ_V associated to $V \subseteq H^0(X,L)$ if $s(x)=0$, $\forall\, s \in V$.

In case $V = H^0(X,L)$, we may call such a point $x \in X$ a base point for L.

The closed subset

$$Bs(L) = \{x \in X \mid s(x) = 0\ \forall\, s \in H^0(X,L)\}$$

is called the base locus of L.

Choosing a $\mathbb{C}$-basis $\{s_0, ---, s_N\} \subseteq H^0(X,L)$, we see that $Bs(L) = \bigcap_{i=0}^{N} Z(s_i)$, where $Z(s_i) = \{x \in X \mid s_i(x) = 0\}$, $\forall\, 0 \le i \le s$.

We say that $|L|$ is base point free if $Bs(L) = \phi$.

Proposition: Let L be a holomorphic line bundle on a compact complex manifold X. Then there is a holomorphic map

$$\varphi_L : X \setminus Bs(L) \longrightarrow \mathbb{P}(H^0(X,L)^\vee) \cong \mathbb{C}\mathbb{P}^N,$$

where $N = \dim_{\mathbb{C}}(H^0(X,L)) - 1$.

Sketch of a proof:- Send $x \in X \setminus Bs(L)$ to the class of $\mathbb{C}$-linear map $\varphi_L(x) : H^0(X,L) \longrightarrow L|_x \cong \mathbb{C}$
$s \longmapsto s(x)$ $\square$

Remark: φ_L is a rational map

$$X \dashrightarrow \mathbb{P}(H^0(X,L)^\vee)$$

which is defined on whole X iff $Bs(L) = \phi$.

Definition: A complex manifold X is said to be projective if there is a closed embedding

$$X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$$

for some integer $N \geq 1$.

Clearly a projective manifold must be compact.

We are interested to put some reasonable condition on X so that X admits a line bundle L with $Bs(L) = \phi$, and that the induced map

$$\varphi_L : X \longrightarrow \mathbb{P}(H^0(X,L)^\vee) \cong \mathbb{C}\mathbb{P}^N$$

become a closed embedding. This is the main content of Kodaira's embedding theorm.

Suppose that $Bs(L) = \phi$ so that we have a holomorphic map

$$\varphi_L : X \longrightarrow \mathbb{P}(H^0(X,L)^\vee) \cong \mathbb{CP}^N.$$

We are interested to see when φ_L is a closed embedding (meaning that,

(i) φ_L is injective, and

(ii) $(d\varphi_L)_x : T_x X \longrightarrow T_{\varphi_L(x)} \mathbb{P}(H^0(X,L)^\vee)$ is injective, $\forall x \in X$.)

Remark 1: Let $x \in X$. Let $\mathcal{I}_{\{x\}} \subseteq \mathcal{O}_X$ be the subsheaf of sections vanishing at x. Let $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field at $x \in X$. Then we have an exact seq$^{\underline{n}}$

$$0 \to \mathcal{I}_{\{x\}} \longrightarrow \mathcal{O}_X \longrightarrow k(x) \longrightarrow 0$$

Tensoring this with a line bundle L, and applying the functor $H^0(X,-)$, we have a map

$$ev_x : H^0(X,L) \longrightarrow L|_x := L \otimes k(x),$$
$$s \longmapsto s(x)$$

which is surjective iff $H^1(X, L \otimes \mathcal{I}_{\{x\}}) = 0$.

Note that $x \in X \setminus Bs(L)$ iff $ev_x : H^0(X,L) \to L|_x$ is surjective.

Definition: We say that $|L| = \mathbb{P}(H^0(X,L))$ separates points if given any two distinct points $x_1, x_2 \in X$, $\exists$ a section $s \in H^0(X,L)$ such that $s(x_1) = 0$ and $s(x_2) \neq 0$.

Note that, $|L|$ separates points iff the induced map $\varphi_L : X \setminus Bs(L) \longrightarrow \mathbb{P}(H^0(X,L)^\vee)$ is injective.

Remark 2: Assume that $Bs(L) = \phi$, so that we have a holomorphic map
$$\varphi_L : X \longrightarrow \mathbb{P}(H^0(X,L)^\vee) \cong \mathbb{CP}^N.$$

Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Following the discussion in Rem. 1, the short exact seq$\underline{\underline{n}}$ of $\mathcal{O}_X$-modules
$$0 \to L \otimes \mathcal{I}_{\{x_1, x_2\}} \longrightarrow L \longrightarrow L|_{x_1} \oplus L|_{x_2} \longrightarrow 0$$
gives us another evalution map
$$ev_{x_1} \oplus ev_{x_2} : H^0(X, L) \longrightarrow L|_{x_1} \oplus L|_{x_2}$$
$$s \longmapsto (s(x_1), s(x_2))$$

Clearly surjectivity of this map is equivalent to the statement that $|L|$ separates points (i.e., φ_L is injective).

Definition:- We say that $|L|$ separates tangent directions if the map $(d\varphi_L)_x : T_x X \to T_{\varphi_L(x)} \mathbb{P}(H^0(X,L)^\vee)$ is injective for all $x \in X$.

We now reformulate this in terms of surjectivity of certain map, which is useful in many purposes.

Let $x \in X$. Since $Bs(L) = \phi$, $\exists$ $s_0 \in H^0(X, L)$ with $s_0(x) \neq 0$. From the short exact seq$\underline{\underline{n}}$
$$0 \to L \otimes \mathcal{I}_{\{x\}} \longrightarrow L \longrightarrow L|_x \longrightarrow 0$$
and its associated long exact seq$\underline{\underline{n}}$ of cohomologies, we note that $H^1(X, L \otimes \mathcal{I}_{\{x\}}) = 0$ and hence we can choose $s_1, \ldots, s_N \in H^0(X, L \otimes \mathcal{I}_{\{x\}}) \subseteq H^0(X, L)$ such that $\{s_0, s_1, \ldots, s_N\}$ is a basis for $H^0(X, L)$.

Setting $t_i = s_i/s_0$, $1 \leq i \leq N$, locally around x, the map $\varphi_L : X \to \mathbb{P}(H^0(X, L)^\vee) \cong \mathbb{CP}^N$ can be given by sending $y \in X$ to $(t_1(y), \ldots, t_N(y)) \in \mathbb{C}^N \subseteq \mathbb{CP}^N$, where $t_1(x) = t_2(x) = \cdots = t_N(x) = 0$.

Then the differential map (at x)

$$(d\varphi_L)_x : T_x X \longrightarrow T_{\varphi_L(x)} \mathbb{CP}^N$$

is injective iff the set of 1-forms $\{dt_1, \ldots, dt_N\}$ spans the cotangent space $T_x^* X = \mathfrak{m}_x/\mathfrak{m}_x^2 \cong \mathcal{I}_{\{x\}}/\mathcal{I}_{\{x\}}^2$.

We have the following short exact seq$\underline{\underline{n}}$.

$$0 \to L \otimes \mathcal{I}_{\{x\}}^2 \longrightarrow L \otimes \mathcal{I}_{\{x\}} \longrightarrow L \otimes T_x^* X \longrightarrow 0$$

Applying the functor $H^0(X, -)$, we get a map

$$d_x : H^0(X, L \otimes \mathcal{I}_{\{x\}}) \longrightarrow L|_x \otimes_{\mathbb{C}} T_x^* X,$$

which can be described in terms of a local trivialization $\psi : L|_U \xrightarrow{\sim} U \times \mathbb{C}$ as

$$s \in H^0(X, L \otimes \mathcal{I}_{\{x\}}) \xmapsto{d_x} d(\psi \circ s)_x \in L|_x \otimes T_x^* X.$$

(One can check that this map is independent of choice of trivialization ψ).

Now continuing with the above discussion, with $t_i = s_i/s_0$, one find that $(dt_i)_x = (\psi s_0)^{-1} d(\psi s_i)_x$, $\forall\, 1 \leq i \leq N$.

Then one can check that, $\{dt_1, \ldots, dt_N\}$ spans $T_x^* X$ iff the above constructed map $d_x : H^0(X, L \otimes \mathcal{I}_{\{x\}}) \to L|_x \otimes T_x^* X$ is surjective.

§ Positive line bundles :-

Definition:- A holomorphic line bundle L on a manifold X is said to be positive if there is a Hermitian metric h on L such the induced real $(1,1)$-form $i\Theta_h \in A^{1,1}_X$ is positive, where Θ_h is the curvature of the Chern connection D_h of the Hermitian metric h on L.

In other words, the curvature Θ_h should satisfy $\Theta_h(v, \bar{v}) > 0$, $\forall v \in T_x X \setminus \{0\}$.

Remark/Proposition: Given any Hermitian metric h on L, the form $-\frac{1}{2\pi i}\Theta_h$ represents the first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ of L. (Since $c_1(L)$ does not depend on the choice of a C^∞ connection on L). We can rephrase the above definition as follow:

A line bundle L on X is said to be positive if its first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ can be represented by a positive closed real $(1,1)$-form on X.

Remark/Proposition: Let X be a compact Kähler manifold. Given any closed real $(1,1)$-form $\alpha \in A_X^{1,1}$ representing the first Chern class $c_1(L) \in H^2(X,\mathbb{Z})$ of a line bundle L on X, there is a Hermitian metric h on L such that

$$\alpha = -\frac{1}{2\pi i}\Theta_h \quad , \text{ where } \Theta_h \text{ is the curvature of the}$$

Chern connection D_h on (L,h).

Proposition:- Tensor product of positive line bundles is positive.

Proof: Let L_1 and L_2 be two positive line bundles on X. Let h_1 and h_2 be two Hermitian metrics on L_1 and L_2 such that the associated Chern connections D_{h_1} and D_{h_2} on L_1 and L_2 has curvatures Θ_1 and Θ_2 which satisfies $\Theta_1(v,\bar{v}) > 0$ & $\Theta_2(v,\bar{v}) > 0$ $\forall\ 0 \neq v \in T_xX$.

Then $D = D_{h_1} \otimes Id + Id \otimes D_{h_2}$ is a C^∞ connection on $L_1 \otimes L_2$ with curvature $\Theta_D = \Theta_1 \otimes Id + Id \otimes \Theta_2$.

Hence the result follows. $\blacksquare$

Definition: A line bundle L on X is called very ample if $Bs(L) = \phi$ and the induced map $\varphi_L : X \longrightarrow \mathbb{P}(H^0(X,L)^\vee)$ is an embedding.

A line bundle L on X is said to be ample if $L^{\otimes n}$ is very ample, for some integer $n \geq 1$.

Proposition :- Any ample line bundle is positive.

Proof :- Let L be an ample line bundle on a complex manifold X. Then $\exists$ an integer $n \geq 1$ such that L^n is very ample, and hence defines an embedding

$$\varphi_{L^n} : X \longrightarrow \mathbb{CP}^N$$

for some $N \gg 1$. Then $L^n \cong \varphi_{L^n}^* \mathcal{O}(1)$, and hence the pullback of the Fubini-Study Kähler form ω_{FS} over X gives the first chern class of $L^{\otimes n}$. Since n is positive integer, the form ω_{FS} is positive, we conclude that L is positive.

Key Lemma : Let L be a positive line bundle on a compact complex manifold X. Fix a finite subset $S = \{x_1, \ldots, x_\ell\}$ of X, and let $\sigma : \tilde{X} = Bl_S(X) \longrightarrow X$ be the blow-up of X along S. Let $E_j = \sigma^{-1}(x_j)$, $1 \leq j \leq \ell$.

Then for a line bundle M on X and integers $n_1, n_2, \ldots, n_\ell > 0$, there is a positive integer $k > 0$ such that the line bundle

$$\sigma^*(L^{\otimes k} \otimes M) \otimes \mathcal{O}_{\tilde{X}}\left(-\sum_{j=1}^{\ell} n_j E_j\right)$$

on $\tilde{X}$ is positive.

Sketch of a proof :- Note that, in a neighbourhood $U_j \subseteq X$ of $x_j \in X$, the blow-up of X at x_j can be constructed as the incidence variety (total space of the tautological line bundle of $\mathbb{P}^{n-1}_{\mathbb{C}}$)

$$\widetilde{U}_j := \mathcal{O}(-1) \subset U_j \times \mathbb{P}^{n-1}_{\mathbb{C}},$$

and the line bundle $\mathcal{O}_{\widetilde{U}_j}(E_j) \cong q_j^* \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(-1)$, where

$q_j : \widetilde{U}_j \longrightarrow \mathbb{P}^{n-1}_{\mathbb{C}}$ is the second projection map.

Then the pull-back of the Fubini-Study Kähler metric from $\mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(1)$ gives a Hermitian metric on $\mathcal{O}(-E_j) \cong q_j^* \mathcal{O}_{\mathbb{P}^{n-1}_{\mathbb{C}}}(1)$, and hence on their tensor powers $\mathcal{O}_{\widetilde{X}}(-n_j E_j) = \mathcal{O}_{\widetilde{X}}(-E_j)^{\otimes n_j}$.

Note that, $\mathcal{O}_{\widetilde{X}}(-E_j)$ is trivial outside $E_j = \sigma^{-1}(x_j)$, $\forall 1 \leq j \leq l$.

Gluing these Hermitian metrics using a partition of unity, we get a Hermitian metric h on $\mathcal{O}_{\widetilde{X}}(-\sum_{j=1}^{l} n_j E_j)$.

Since $\mathcal{O}(-E_j) \cong q_j^* \mathcal{O}(1)$ by construction, locally around $E_j = \sigma^{-1}(x_j)$, the curvature Θ_h of the Chern connection D_h on $\mathcal{O}_{\widetilde{X}}(-\sum_{j=1}^{l} n_j E_j)$ is of the form $-2\pi i n_j q_j^* \omega_{FS}$, where ω_{FS} is the Fubini-Study Kähler form on $\mathbb{P}^{n-1}_{\mathbb{C}}$.

(This is because, the curvature of the Chern connection of the Fubini-study hermitian metric on $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ is $-2\pi i\, \omega_{FS}$, and curvature is compatible with pull-back and tensor product).

Thus the curvature Θ_h is semi-positive over $\widetilde{X}$ and is strictly positive over each E_j, $1 \leq j \leq l$.

Now look at the line bundle

$$\widetilde{L} := \sigma^*(L^{\otimes k} \otimes M) \otimes \mathcal{O}_{\widetilde{X}}(-\sum_{j=1}^{l} n_j E_j) \quad \text{on } \widetilde{X}.$$

Since L is positive, $c_1(L) \in H^2(X, \mathbb{Z})$ is given by a positive closed real $(1,1)$-form α on X. Let $c_1(M) = [\beta]$, for some closed real $(1,1)$-form β on X.

Then $c_1(\tilde{L})$ is represented by the form

$$\sigma^*(k\cdot\alpha + \beta) + \left(\frac{i}{2\pi}\right)\Theta_h,$$

which is positive for $k \gg 0$. This completes the proof. ▨

Theorem (Kodaira's Embedding Theorem):-

Let X be a compact Kähler manifold. Then X is projective (i.e., a closed submanifold of a complex projective space $\mathbb{CP}^N$) if and only if there is a positive line bundle on X.

Proof:- We already have seen that any ample line bundle on a complex manifold is positive. So if X is projective, then there is a closed embedding $X \xhookrightarrow{i} \mathbb{P}^N_{\mathbb{C}}$, for some integer $N \geq 1$, and then $i^*\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(1)$ is an ample (in fact, very ample) line bundle on X, and hence is positive.

Conversely, suppose that X admits a positive line bundle, say L. To show X projective, it suffices to show that for some integer $m \gg 1$, the line bundle $L^{\otimes m}$ defines a closed embedding

$$\varphi_{L^m} : X \longrightarrow \mathbb{P}(H^0(X, L^{\otimes m})^\vee) \cong \mathbb{CP}^N.$$

Step 1: We first show that $Bs(L^{\otimes m}) = \phi$, for some $m \gg 1$, so that the map φ_{L^m} is defined on whole X.

Here we need to use the fact that X is compact and Kähler.

Step 2: For $m \gg 1$, the map φ_{L^m} separates points and tangent directions, and hence is a closed embedding of X into a complex projective space.

Proof of Step-1: We need to show the following:

Claim 1: Given any point $x \in X$, $\exists$ an integer $m_x \geq 1$ such that $x \notin Bs(L^m)$, $\forall\, m \geq m_x$.

As an immediate consequence to Claim 1, we have the following:

Corollary to Claim 1: There is a positive integer $m_0 \geq 1$ such that $Bs(L^m) = \phi$, $\forall\, m \geq m_0$.

Proof: Since the base field $\mathbb{C}$ has characteristic 0, for each $i \geq 1$, considering the map

$$H^0(X, L^{2^i}) \longrightarrow H^0(X, L^{2^i} \otimes L^{2^i}) = H^0(X, L^{2^{i+1}})$$
$$s \longmapsto s \otimes s$$

we see that $Bs(L^{2^{i+1}}) \subseteq Bs(L^{2^i})$. So we have a decreasing seq^{n}. of closed subsets

$$Bs(L) \supseteq Bs(L^2) \supseteq Bs(L^{2^2}) \supseteq Bs(L^{2^3}) \supseteq Bs(L^{2^4}) \supseteq \cdots$$

Since L is positive, so is L^m, $\forall\, m \geq 1$.
Since X is compact, it follows from the above Claim-1 that $\exists$ an integer $m_0 \geq 1$ s.t. $Bs(L^m) = \phi$ $\forall\, m \geq m_0$.
(Note: $\{X \setminus Bs(L^{m_x})\}_{x \in X}$ is an open cover of X).

Claim-1: Given any point $x \in X$, $\exists$ an integer $m_x \geq 1$ such that $x \notin Bs(L^m)$, $\forall\, m \geq m_x$.

Proof of claim 1: Recall that, a point $x \in X \setminus Bs(L)$ if and only if the evaluation map

$$ev_x : H^0(X, L) \longrightarrow L|_x$$
$$s \longmapsto s(x)$$

is surjective. To achieve this in our case, we use blow-up. Let

$$\sigma : \tilde{X} = Bl_x(X) \longrightarrow X$$

be the blow-up of X at $x \in X$, and let $E = \sigma^{-1}(x) \cong \mathbb{P}^{n-1}_{\mathbb{C}}$ be the exceptional divisor. Note that $H^0(E, \mathcal{O}_E) \cong \mathbb{C}$.

Let $\iota_x : \{x\} \hookrightarrow X$ be the inclusion map.

Then we have the following commutative diagram

$$\begin{tikzcd}
s \in H^0(X, L^m) \arrow[r, "s \,\mapsto\, s(x)"] \arrow[d, "\tilde{\sigma}"] & \iota_x^* L^m = L^m|_x \arrow[d, "\cong"] \\
\sigma^* s \in H^0(\tilde{X}, \sigma^* L^m) \arrow[r] & \underbrace{H^0(E, \mathcal{O}_E)}_{\cong \mathbb{C}} \otimes \iota_x^* L^m
\end{tikzcd}$$

Since the blow-up map $\sigma : \tilde{X} \longrightarrow X$ is surjective, the left vertical map $s \overset{\tilde{\sigma}}{\longmapsto} \sigma^* s$ is injective. We show that this map is, in fact, an isomorphism.

If $\dim_{\mathbb{C}}(X)=1$, the blow-up map is an iso, and so is $\tilde{\sigma}$.

Assume that $\dim_{\mathbb{C}}(X)=n\geqslant 2$. Let $s\in H^0(\tilde{X},\sigma^*L^m)$.

Since
$$\sigma' := \sigma|_{\tilde{X}\setminus E} : \tilde{X}\setminus E \xrightarrow{\cong} X\setminus\{x\}$$
is an isomorphism, and $\operatorname{codim}_X(\{x\})\geqslant 2$, by Hartog's extension theorm the section $\sigma'_*(s|_{\tilde{X}\setminus E})\in H^0(X\setminus\{x\},L^m)$ extends to a section $\tilde{s}\in H^0(X,L^m)$.

Therefore, $\tilde{\sigma}: H^0(X,L^m)\longrightarrow H^0(\tilde{X},\sigma^*L^m)$ is surjective, and hence is an isomorphism.

To show the map $ev_x : H^0(X,L^m)\longrightarrow L^m|_x$ is surjective for $m>>1$, from the above commutative diagram, it suffices to show that the cokernel of the map
$$H^0(\tilde{X},\sigma^*L^m)\longrightarrow H^0(E,\mathcal{O}_E)\otimes L^m|_x$$
vanishes, for $m>>1$. Here we need Kähler structure on $\tilde{X}$ (which we get from the Kähler structure on X) and positivity of L to use Kodaira vanishing theorem.

It follows from the short exact seq^n:
$$0\to \sigma^*L^m\otimes\mathcal{O}_{\tilde{X}}(-E)\longrightarrow \sigma^*L^m\longrightarrow \sigma^*L^m|_E\cong L^m|_x\otimes\mathcal{O}_E\to 0$$
that
$$\operatorname{coker}\left(H^0(\tilde{X},\sigma^*L^m)\to H^0(E,\mathcal{O}_E)\otimes L^m|_x\right)\subseteq H^1(\tilde{X},\sigma^*L^m\otimes\mathcal{O}_{\tilde{X}}(-E)).$$

Since $K_{\tilde{X}} \cong \sigma^* K_X \otimes \mathcal{O}_{\tilde{X}}((n-1)E)$, by above key Lemma, (with $M = K_X^\vee$) the line bundle

$$L'_m := \sigma^* L^m \otimes K_{\tilde{X}}^\vee \otimes \mathcal{O}_{\tilde{X}}(-E)$$
$$\cong \sigma^* L^m \otimes \sigma^* K_X^\vee \otimes \mathcal{O}_{\tilde{X}}(-nE)$$
$$\cong \sigma^*(L^m \otimes K_X^\vee) \otimes \mathcal{O}_{\tilde{X}}(-nE)$$

$\Big| -(n-1)-1 = -n$

is positive, for $m >> 1$.

Since X is compact and Kähler, so is its blow-up $\tilde{X}$.

So by Kodaira vanishing theorem, we have

$$H^1(\tilde{X}, \sigma^* L^m \otimes \mathcal{O}_{\tilde{X}}(-E)) \cong H^1(\tilde{X}, K_{\tilde{X}} \otimes L'_m) = 0$$

Kodaira's Vanishing theorem: Let X be a compact Kähler manifold of dimension n. Then for any positive line bundle L on X, we have

$$H^q(X, \Omega_X^p \otimes L) = 0, \text{ whenever } p+q > n.$$

Therefore the map

$$ev_x : H^0(X, L^m) \longrightarrow L^m|_x$$

is surjective, for $m >> 1$. So $x \notin Bs(L^m)$ $\forall\, m \geq m_x$.

Then by the above Corollary to Claim 1, $Bs(L^m) = \phi$ for $m >> 1$.

This completes the proof of Step-1.

Step-2: For $m >> 1$, the map $\varphi_{L^m} : X \longrightarrow \mathbb{P}(H^0(X, L^m)^\vee) \cong \mathbb{CP}^N$ separates points and tangent directions (and hence is a closed embedding).

Proof of Step-2:

Given any two distinct points $x_1, x_2 \in X$, using a similar arguments described above, working with the line bundle $\sigma^* L^m \otimes \mathcal{O}_{\tilde{X}}(-E_1 - E_2)$, one can conclude that the map

$$ev_{x_1} \oplus ev_{x_2} : H^0(X, L^m) \longrightarrow L^m|_{x_1} \oplus L^m|_{x_2}$$

is surjective, for $m >> 1$. In other words, the map $\varphi_{L^m} : X \longrightarrow \mathbb{P}(H^0(X, L^m)^\vee) \cong \mathbb{CP}^N$ is injective, for $m >> 1$.

To show that $|L^m|$ separates tangent directions, for $m >> 1$, let $x \in X$, and consider the exact sequences

$$0 \to \mathcal{I}^2_{\{x\}} \longrightarrow \mathcal{I}_{\{x\}} \longrightarrow \mathcal{I}_{\{x\}} / \mathcal{I}^2_{\{x\}} \cong T^*_x X \longrightarrow 0$$

and $$0 \to \mathcal{O}_{\tilde{X}}(-2E) \to \mathcal{O}_{\tilde{X}}(-E) \to \mathcal{O}_{\tilde{X}}(-E)|_E \cong \mathcal{O}(1) \longrightarrow 0$$

where $\mathbb{P}^{n-1}_{\mathbb{C}} \cong E := \sigma^{-1}(x) \subset \tilde{X}$ is the exceptional divisor over x.

Now pulling back sections of $\mathcal{I}_{\{x\}}$ and $\mathcal{I}^2_{\{x\}}$ along the blow-up map $\sigma : \tilde{X} = Bl_x(X) \longrightarrow X$, we have the following commutative diagram of $\mathcal{O}_{\tilde{X}}$-modules.

$$\begin{array}{ccc} \sigma^* \mathcal{I}^2_{\{x\}} & \longrightarrow & \sigma^* \mathcal{I}_{\{x\}} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\tilde{X}}(-2E) & \longrightarrow & \mathcal{O}_{\tilde{X}}(-E) \end{array} .$$

(diagram-1)

Tensoring this diagram with $L^{\otimes m}$, passing to the quotients and applying $H^0(\tilde{X}, -)$, as before, we get the following commutative diagram.

$$\begin{array}{ccc} H^0(X, L^m \otimes \mathcal{I}_{\{x\}}) & \xrightarrow{\Phi} & L^m|_x \otimes T_x^* X \\ \downarrow & & \downarrow \\ H^0(\tilde{X}, \sigma^* L^m \otimes \mathcal{O}_{\tilde{X}}(-E)) & \longrightarrow & L^m|_x \otimes H^0(E, \mathcal{O}_{\tilde{X}}(-E)|_E) \end{array}$$

Following the similar arguments as in Step 1, one can conclude that the vertical arrow on the left

$$H^0(X, L^m \otimes \mathcal{I}_{\{x\}}) \xrightarrow{\cong} H^0(\tilde{X}, \sigma^* L^m \otimes \mathcal{O}_{\tilde{X}}(-E))$$

is an iso. Since $N_{\{x\}/X} = T_x^* X$ gives $E \cong \mathbb{P}(T_x^* X)$, and we know that $\mathcal{O}_{\tilde{X}}(-E)|_E \cong \mathcal{O}(1)$, we see that $H^0(E, \mathcal{O}_{\tilde{X}}(-E)|_E) \cong T_x^* X$.

Note that, the vertical map on the right hand side is induced by the map (see diagram-1)

$$\mathcal{O}_E \otimes T_x^* X \longrightarrow \mathcal{O}_{\tilde{X}}(-E)|_E,$$

which is actually an evaluation map $\mathcal{O}_E^{\oplus n} \twoheadrightarrow \mathcal{O}_E(1)$, and hence surjective. Therefore, the vertical map on the right hand side of the above diagram is surjective, and hence is an isomorphism of $\mathbb{C}$-vector spaces.

Then as shown in Step-1, to show Φ is surjective, it is enough to show that $H^1(\tilde{X}, \sigma^* L^m \otimes \mathcal{O}_{\tilde{X}}(-2E)) = 0$. As before, this cohomology vanishing follows from Kodaira vanishing theorem.

This completes the proof. $\square$

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