EQUIVARIANT PARABOLIC CONNECTIONS AND STACK OF ROOTS

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ABSTRACT. Let X be a smooth complex projective variety equipped with an action of a linear algebraic group G over $\mathbb{C}$. Let D be a reduced effective divisor on X that is invariant under the G–action on X. Let s_D be the canonical section of $\mathcal{O}_X(D)$ vanishing along D. Given a positive integer r, consider the stack $\mathfrak{X} := \mathfrak{X}_{(\mathcal{O}_X(D), s_D, r)}$ of r-th roots of $(\mathcal{O}_X, s_D)$ together with the natural morphism $\pi : \mathfrak{X} \to X$. Under the assumption that G has no non-trivial characters, we show that the G–action on X naturally lifts to a G–action on $\mathfrak{X}$ such that π become G–equivariant, and the tautological invertible sheaf $\mathscr{M}$ on $\mathfrak{X}$ admits a linearization of this G–action. Finally, we define the notions of G–equivariant logarithmic connections on $\mathfrak{X}$ and G–equivariant parabolic connections on X with rational parabolic weights along D, and establish an equivalence between the category of G–equivariant logarithmic connections on $\mathfrak{X}$ and the category of G–equivariant parabolic connections on X with rational parabolic weights along D.

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1. INTRODUCTION

The notion of parabolic vector bundles on a compact Riemann surface X together with a finite subset $D \subset X$ of marked points is introduced in [MS80] by V. Mehta and C.S. Seshadri. In their initial formulation, a parabolic bundle is a vector bundle E on X equipped with the data of a flag $F_\bullet(E)$ of $\mathbb{C}$–linear subspaces of each of its fibers E_x over the points $x \in D$, and a set of increasing real numbers in $[0, 1)$ associated to each points of D, called weights. The celebrated Mehta–Seshadri theorem establishes a one-to-one correspondence between polystable parabolic vector bundles on (X, D) of parabolic degree zero and the unitary representations of the fundamental group $\pi_1(X \setminus D)$ of the punctured Riemann surface $X \setminus D$. Later Simpson reformulated and extended the notion of parabolic vector bundles to the case of schemes X with a normal crossings divisor D in [Sim90]. When the weights are all rational, and therefore can be assumed to be in $\frac{1}{r}\mathbb{Z}$ for some integer r, this amounts to a locally free coherent sheaf $\mathcal{E}$ of finite rank, together with a filtration

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_{\frac{1}{r}} \supset \mathcal{E}_{\frac{2}{r}} \supset \cdots \supset \mathcal{E}_{\frac{r-1}{r}} \supset \mathcal{E}_1 = \mathcal{E}(-D)$$

by its coherent subsheaves [MY92, Bis97]. Since their introduction, parabolic bundles, parabolic connections and their moduli spaces have been an active area of research in algebraic geometry and differential geometry, and have been used in a number of important applications. Even though most of the progress has been done in the case of curves, more recently, parabolic connections have made appearance in the work of Donagi and Pantev on Geometric Langlands conjecture using Simpson's non-abelian Hodge theory [DP22].

When the weights are rational, it is known that parabolic bundles can be interpreted equivalently as equivariant bundles on a suitable ramified Galois cover [Bis97]. A more intrinsic interpretation is found later using vector bundles on certain algebraic stacks associated to (X, D), namely the stack of roots in [Bor07]. More precisely, there is a Fourier like correspondence between parabolic vector bundles on a scheme and ordinary vector bundles on certain stack of roots. This naturally raised the question of understanding the parabolic connections on smooth varieties through such Fourier like correspondence over root stacks. This has been shown to be true over curves [BMW12, LSS13], and very recently over higher dimensional varieties in [BL23]. This correspondence has been further extended to the case of real parabolic connections on a real variety (X, D) in [CP24], and for orthogonal and symplectic parabolic connections in [CM24].

In this article we study such correspondences under the presence of a group action. More precisely, let X be a connected smooth complex projective variety. Let G be a connected complex linear algebraic group acting on X. Let D be a smooth reduced effective Cartier divisor on

X invariant under the G–action on X. Given a positive integer r, we can associate a stack of r-th roots of $\mathcal{O}_X(D)$ and its canonical section vanishing along D, which we denote by $\mathfrak{X}$. It comes with a natural morphism $\pi : \mathfrak{X} \to X$. The root stack $\mathfrak{X}$ admits a tautological invertible sheaf $\mathcal{M}$ satisfying $\pi^*\mathcal{O}_X(D) \simeq \mathcal{M}^{\otimes r}$. It is a natural to ask whether the G-action on X can be lifted to a G-action on $\mathfrak{X}$ making the morphism $\pi : \mathcal{X} \to X$ G–equivariant. We address this question first; we show that, if G has no non-trivial characters, then the G–action on (X, D) can be lifted to $\mathfrak{X}$ making π G-equivariant. Moreover, the tautological invertible sheaf $\mathcal{M}$ naturally admits a linearization for this G-action on $\mathfrak{X}$. This is done in Proposition 2.2.2 and Proposition 2.3.11.

It is shown in [Bor07] that there is an equivalence between the category of algebraic vector bundles on X and the category of parabolic vector bundles on X with parabolic structures along D and having rational weights in $\frac{1}{r}\mathbb{Z}$. We extend this result to the case of G-equivariant parabolic bundles (see Definition 3.2.2).

Theorem 1.0.1 (Theorem 3.2.4). *There is an equivalence between the category of G-equivariant vector bundles on $\mathfrak{X}$ and the category of G-equivariant parabolic bundles on X with parabolic structures along D having rational weights in $\frac{1}{r}\mathbb{Z}$.*

Finally, let $\mathfrak{D}$ be an effective Cartier divisor on $\mathfrak{X}$ satisfying $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \simeq \mathcal{M}$. The correspondence of N. Borne as above has been recently generalized to an equivalence between the category of logarithmic connections on $\mathfrak{X}$ along $\mathfrak{D}$ and the category of parabolic connections on X along D and weights in $\frac{1}{r}$ when D has strict normal crossings [BL23]. In the final section, we consider D to be a smooth irreducible effective Cartier divisor on X, and extend the result of [BL23] to our equivariant setup.

Theorem 1.0.2 (Theorem 4.4.2). *There is an equivalence between the category of G-equivariant logarithmic connections on $(\mathfrak{X}, \mathfrak{D})$ (Definition 4.3.2) and the category of G-equivariant parabolic connections on (X, D) (Definition 4.2.2).*

2. LIFTING G-ACTION TO ROOT STACKS

Unless explicitly specified, all the stacks in this article are algebraic and are defined over $\mathbb{C}$. We use the same symbol T to denote a $\mathbb{C}$-scheme T as well as the $\mathbb{C}$-stack represented by the scheme T, if no confusion arises. Given $\mathbb{C}$-stacks $\mathcal{Y}$ and $\mathcal{Y}'$, we use the symbol $\mathcal{Y} \times \mathcal{Y}'$ to mean their fiber product $\mathcal{Y} \times_{\mathrm{Spec}\,\mathbb{C}} \mathcal{Y}'$ over $(\mathrm{Sch}\,/\mathbb{C})$; note that it is a $\mathbb{C}$-stack and for any $T \in (\mathrm{Sch}\,/\mathbb{C})$ we have

$$(\mathcal{Y} \times \mathcal{Y}')(T) = \mathcal{Y}(T) \times \mathcal{Y}'(T), \tag{2.0.1}$$

because $(\mathrm{Spec}\,\mathbb{C})(T)$ is singleton (c.f. [CMW18, §4.1] or [Ols07]).

2.1. **Preliminaries on root stack.** We recall some basic definition and properties of root stack from [Cad07]. A *generalized Cartier divisor* on a scheme T is an ordered pair (M, s), where M is an invertible sheaf of $\mathcal{O}_T$-modules on T and $s \in \Gamma(T, M)$. Since a morphism $X \to [\mathbb{A}^1/\mathbb{G}_m]$ is given by a principal $\mathbb{G}_m$-bundle on X equipped with a section of the associated $\mathbb{A}^1$–bundle on X, and vice versa, to give a generalized Cartier divisor (L, s) on X is equivalent to give a

morphism into the quotient stack

$$f_{(L,s)} : X \to [\mathbb{A}^1/\mathbb{G}_m].$$

Given an integer $r \geq 2$, let $\theta_r : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism of quotient stacks induced by the r-th power maps on both $\mathbb{A}^1$ and $\mathbb{G}_m$. Then the fiber product

$$\mathfrak{X}_{(L,s,r)} := X \times_{f_{(L,s)}, [\mathbb{A}^1/\mathbb{G}_m], \theta_r} [\mathbb{A}^1/\mathbb{G}_m]$$

is a Deligne-Mumford stack over $\mathbb{C}$, called the *stack of r-th roots of* (L, s) on X, [Cad07, Theorem 2.3.3]. It has the following description: the objects of $\mathfrak{X}_{(L,s,r)}$ are given by the quadruples $(f : U \to X, M, \phi, t)$, where

- $f : U \to X$ is a morphism of $\mathbb{C}$-schemes,

- M is an invertible sheaf of $\mathcal{O}_U$-modules on U,

- $t \in \Gamma(U, M)$, and

- $\phi : M^{\otimes r} \xrightarrow{\simeq} f^*L$ is an $\mathcal{O}_U$-module isomorphism such that $\phi(t^{\otimes r}) = f^*s$.

A morphism from $(f : U \to X, M, \phi, t)$ to $(f' : U' \to X, M', \phi', t')$ in $\mathfrak{X}_{(L,s,r)}$ is given by a pair (g, ψ), where

- $g : U \to U'$ is a morphism of X-schemes, and

- $\psi : M \to g^*M'$ is an $\mathcal{O}_U$-module isomorphism such that $\psi(t) = g^*(t')$.

Then we have a natural morphism of stacks

$$\pi : \mathfrak{X}_{(L,s,r)} \to X \tag{2.1.1}$$

that sends the object $(f : U \to X, M, \phi, t)$ of $\mathfrak{X}_{(L,s,r)}$ to the X-scheme $f : U \to X$, and a morphism (g, ψ), as above, to the morphism of X-schemes $g : U \to U'$. Note that there is a natural morphism $X \to \operatorname{Spec} \mathbb{C}$ (arising from the structure morphism of X), whose composition with π makes $\mathfrak{X}_{(L,s,r)}$ into a $\mathbb{C}$-stack.

The projection map onto the second factor

$$X \times_{f_{(L,s)}, [\mathbb{A}^1/\mathbb{G}_m], \theta_r} [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$$

corresponds to the *tautological generalized Cartier divisor* $(\mathscr{M}, \zeta)$ on $\mathfrak{X}_{(L,s,r)}$, where $\mathscr{M}$ is the *tautological invertible sheaf* on $\mathfrak{X}$ and ζ is the *tautological global section* of $\mathscr{M}$. Note that $\pi^*(L) \cong \mathscr{M}^{\otimes r}$.

2.2. **G-action on a root stack.** Let G be a connected affine algebraic group over $\mathbb{C}$ acting on a smooth complex projective variety X by $\sigma : G \times X \to X$. Let $p_2 : G \times X \to X$ be the projection map onto the second factor. Recall that a *linearization* of the G-action σ on X to an invertible sheaf L on X is an isomorphism

$$\Phi : p_2^*L \xrightarrow{\simeq} \sigma^*L$$

of sheaves of $\mathcal{O}_{G \times X}$-modules on $G \times X$ satisfying the following cocycle condition coming from compatibility with the G-action on X:

$$(\mu \times \operatorname{Id}_X)^*\Phi = (\operatorname{Id}_G \times \sigma)^*\Phi \circ p_{23}^*\Phi,$$

where $p_{23} : G \times G \times X \to G \times X$ is the projection morphism onto the second and third factors, and $\mu : G \times G \to G$ is the group operation on G (see [MFK94]). For G connected, the above cocycle condition is automatic for an invertible sheaf L on X, see [Dol03, §7.2, Lemma 7.1].

Let D be an effective Cartier divisor on X which is invariant under the G-action, in the sense that $\sigma^*(D) = p_2^*(D)$. Note that, this implies, in particular, that $g^*D = D$, $\forall\, g \in G$, under the natural identification $\{g\} \times X \simeq X$. Let $s_D \in H^0(X, \mathcal{O}_X(D))$ be the canonical section of the invertible sheaf $\mathcal{O}_X(D)$ on X whose divisor of zeros $Z(s_D)$ equals D (cf. [Har77, p. 157]). Assume that G has no non-trivial characters. This is the case, for example, when $G = [G, G]$. For example, any connected semisimple algebraic group G over $\mathbb{C}$ satisfies $G = [G, G]$; and standard examples of such groups include $\mathrm{SL}_n(\mathbb{C}), \mathrm{SO}_n(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C})$ etc.

Lemma 2.2.1. *Suppose that G has no non-trivial characters. Then with the above assumptions, there is a natural G–linearization*

$$\Phi : p_2^* \mathcal{O}_X(D) \xrightarrow{\simeq} \sigma^* \mathcal{O}_X(D)$$

of σ on $\mathcal{O}_X(D)$ such that $\Phi(p_2^* s_D) = \sigma^* s_D$. Consequently, $\mathcal{O}_X(nD)$ admits a natural G–linearization Φ_n, for all $n \in \mathbb{Z}$.

Proof. The assumption $p_2^*D = \sigma^*D$ gives rise to a natural isomorphism

$$\Phi : p_2^* \mathcal{O}_X(D) \xrightarrow{\simeq} \sigma^* \mathcal{O}_X(D)$$

of invertible sheaves on $G \times X$. We need to show that $\Phi(p_2^* s_D) = \sigma^* s_D$. For each $g \in G$, the natural identification $\{g\} \times X \simeq X$ gives rise to an isomorphism of invertible sheaves

$$\Phi_g : \mathcal{O}_X(D) \xrightarrow{\simeq} \sigma_g^* \mathcal{O}_X(D)$$

on X, where $\sigma_g : X \to X$ is the multiplication by g map. Note that, the sections $\Phi_g(s_D)$ and $\sigma_g^*(s_D)$ of $\sigma_g^* \mathcal{O}_X(D)$ satisfies

$$Z(\Phi_g(s_D)) = Z(s_D) = D \text{ and } Z(\sigma_g^*(s_D)) = \sigma_g^*(D).$$

Since $\sigma_g^*(D) = D$, we see that the sections $\Phi_g(s_D)$ and $\sigma_g^*(s_D)$ have the same divisor of zeros. Since X is a smooth projective variety, there exists a $\lambda_g \in \mathbb{C}^*$ such that $\Phi_g(s_D) = \lambda_g \cdot \sigma_g^*(s_D)$ (see [Har77, Proposition 7.7]). This gives a map $\chi : G \to \mathbb{C}^*$ defined by $g \mapsto \lambda_g$. It follows from the cocycle condition for the G–linearization Φ on $\mathcal{O}_X(D)$ that χ is a group homomorphism. Since G has no non-trivial characters by assumption, we conclude that χ is trivial, i.e. $\lambda_g = 1$, $\forall\, g \in G$. From this, the result follows (see [Dol03, p. 105] and [MFK94, p. 32]). $\square$

Henceforth we always assume that G has no non-trivial characters, and $\mathfrak{X} := \mathfrak{X}_{(\mathcal{O}_X(D), s_D, r)}$ be the stack of r-th roots of the generalized Cartier divisor $(\mathcal{O}_X(D), s_D)$ on X. Let $\pi : \mathfrak{X} \to X$ be the natural morphism of stacks over $\mathbb{C}$ as defined in (2.1.1).

Proposition 2.2.2. *With the above assumptions, the G-action σ on X lifts to a G-action*

$$\sigma_{\mathfrak{X}} : G \times \mathfrak{X} \to \mathfrak{X}$$

on $\mathfrak{X}$ (c.f. [Rom05]*) such that the diagram*

$$
\begin{array}{ccc}
G \times \mathfrak{X} & \xrightarrow{\;\sigma_{\mathfrak{X}}\;} & \mathfrak{X} \\
{\scriptstyle \mathrm{Id} \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\
G \times X & \xrightarrow{\;\sigma\;} & X
\end{array}
\qquad (2.2.3)
$$

is 2-*Cartesian.*

Proof. Thinking of both $G \times \mathfrak{X}$ and $\mathfrak{X}$ as fibered categories over (Sch /$\mathbb{C}$), we first describe where $\sigma_{\mathfrak{X}}$ sends the objects and morphisms of $G \times \mathfrak{X}$. For $T \in$ (Sch /$\mathbb{C}$), a typical object of $(G \times \mathfrak{X})(T)$ is given by an ordered pair (g, τ), where $g \in G(T)$ and $\tau = (f : T \to X, M, \varphi, t) \in \mathfrak{X}(T)$, see (2.0.1) and §2.1. Let $h := (g, f) : T \to G \times X$. Then we have

$$
p_1 \circ h = g \ \text{ and } \ p_2 \circ h = f,
$$

where $p_1 : G \times X \to G$ and $p_2 : G \times X \to X$ are the projection maps onto the first and the second factors, respectively. Consider the composite isomorphism of invertible sheaves on T

$$
M^{\otimes r} \xrightarrow{\;\varphi\;} f^* \mathcal{O}_X(D) = h^*(p_2^* \mathcal{O}_X(D)) \xrightarrow{\;h^*(\Phi)\;} h^*(\sigma^* \mathcal{O}_X(D)) = (\sigma \circ h)^* \mathcal{O}_X(D),
$$

where by equality we mean the canonical identification. Note that $(h^*\Phi) \circ \varphi$ sends the section $t^r \in \Gamma(T, M^{\otimes r})$ to

$$
\begin{aligned}
\left((h^*\Phi) \circ \varphi\right)(t^r) = (h^*\Phi)(f^* s_D) &= (h^*\Phi)(h^*(p_2^* s_D)) \\
&= h^*(\Phi(p_2^* s_D)) \\
&= (\sigma \circ h)^* s_D,
\end{aligned}
$$

where the last equality follows from Lemma 2.2.1. Therefore, the quadruple

$$
(\sigma \circ h : T \to X, \ M, \ (h^*\Phi) \circ \varphi, \ t)
$$

defines an object in $\mathfrak{X}(T)$. Therefore, we define

$$
\sigma_{\mathfrak{X}}(g, \tau) := (\sigma \circ h : T \to X, \ M, \ (h^*\Phi) \circ \varphi, \ t) \in \mathfrak{X}(T). \qquad (2.2.4)
$$

We now describe the effect of $\sigma_{\mathfrak{X}}$ on the arrows. Let (g, τ) and (g', τ') be two objects in $G \times \mathfrak{X}$. Then $g \in G(T)$ and $g' \in G(T')$, for some $\mathbb{C}$-schemes T and T', respectively, and that

$$
\tau := (f : T \to X, M, \varphi, t) \ \text{ and } \ \tau' := (f' : T' \to X, M', \varphi', t'), \qquad (2.2.5)
$$

see §2.1. Then a morphism from (g, τ) to (g', τ') in $G \times \mathfrak{X}$ is given by a triple

$$
(\phi, m, (\ell, \theta)) \qquad (2.2.6)
$$

where

- $\phi : T \to T'$ is a morphism of $\mathbb{C}$-schemes,

- $m : T \to T'$ is a morphism of $\mathbb{C}$-schemes making the following diagram commutative

$$
\begin{array}{ccc}
T & \xrightarrow{\;m\;} & T' \\
{\scriptstyle g} \searrow & & \swarrow {\scriptstyle g'} \\
& G &
\end{array}
\qquad (2.2.7)
$$

- $(\ell, \theta) : \tau \to \tau'$ is a morphism in $\mathfrak{X}$; i.e., $\ell : T \to T'$ is a morphism of $\mathbb{C}$-schemes making the following diagram commutative

$$
\begin{array}{ccc}
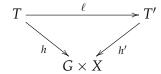T & \xrightarrow{\quad \ell \quad} & T' \\
& \searrow{\scriptstyle f} \quad \swarrow{\scriptstyle f'} & \\
& X &
\end{array}
\tag{2.2.8}
$$

and $\theta : M \longrightarrow \ell^*(M')$ is an isomorphism of $\mathcal{O}_T$-modules such that $\theta(t) = \ell^*(t')$.

Since both m and (ℓ, θ) maps to ϕ under the morphism $\pi : \mathfrak{X} \to X$, it follows that $m = \ell$ in the triple (2.2.6) (c.f. [CMW18, Definition 4.2]). Consider the morphisms

$$
h := (g, f) : T \to G \times X \quad \text{and} \quad h' := (g', f') : T' \to G \times X .
$$

It follows from the commutativity of the diagrams (2.2.7) and (2.2.8) and the equality $\ell = m$ that the following diagram commutes.

$$
\begin{array}{ccc}
T & \xrightarrow{\quad \ell \quad} & T' \\
& \searrow{\scriptstyle h} \quad \swarrow{\scriptstyle h'} & \\
& G \times X &
\end{array}
$$

Therefore, $\sigma \circ h = (\sigma \circ h') \circ \ell$. Thus we get a morphism in $\mathfrak{X}$ given by

$$
\sigma_{\mathfrak{X}}(g, \tau) = (\sigma \circ h, M, (h^*\Phi) \circ \varphi, t) \xrightarrow{(\ell,\theta)} (\sigma \circ h', M', (h'^*\Phi) \circ \varphi', t') = \sigma_{\mathfrak{X}}(g', \tau'),
$$

which we define to be $\sigma_{\mathfrak{X}}((\phi, m, (\ell, \theta)))$. Now it is straight-forward to check that $\sigma_{\mathfrak{X}}$ is indeed a morphism of stacks making the diagram (2.2.3) 2-commutative.

It remains to show that the diagram (2.2.3) is 2-Cartesian. The objects of $(G \times X) \underset{\sigma, X, \pi}{\times} \mathfrak{X}$ are given by quadruples (T, h, τ, α), where

- T is a $\mathbb{C}$-scheme,
- $h \in (G \times X)(T) = \mathrm{Hom}_{(\mathrm{Sch}/\mathbb{C})}(T, G \times X)$,
- $\tau \in \mathfrak{X}(T)$, and
- $\alpha : \sigma \circ h \xrightarrow{\simeq} \pi \circ \tau$ in $X(T) = \mathrm{Hom}_{(\mathrm{Sch}/\mathbb{C})}(T, X)$.

Since $X(T)$ is a set, it follows that α is identity, i.e., $\sigma \circ h = \pi \circ \tau$. Thus the quadruple (T, h, τ, α) simplifies to (T, h, τ) satisfying

$$
\sigma \circ h = \pi \circ \tau.
\tag{2.2.9}
$$

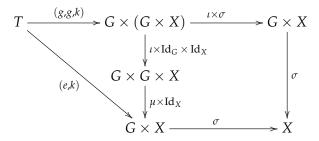If $h = (g, k)$, for some $g : T \to G$, $k : T \to X$, and if $\tau = (f, M, \varphi, t)$ (see §2.1), denoting by $\iota : G \to G$ the inversion morphism of G, the above relation (2.2.9) gives

$$
\begin{aligned}
\sigma \circ (g, k) &= f \\
\implies \sigma \circ (\iota \circ g, \sigma \circ (g, k)) &= \sigma \circ (\iota \circ g, f), \\
\implies k &= \sigma \circ (\iota \circ g, f),
\end{aligned}
$$

where the last equality follows from the following commutative diagram:

$$
\begin{array}{ccc}
T \xrightarrow{\ (g,g,k)\ } G \times (G \times X) \xrightarrow{\ \iota \times \sigma\ } G \times X \\
\end{array}
$$

where $\mu : G \times G \to G$ is the group multiplication, and $e : T \to G$ is the constant map sending everything to the neutral point of G. Thus we can conclude that the data of (T, g, τ) determines map h, and thus there is an isomorphism of fibered categories

$$
\left((G \times X) \underset{\sigma, X, \pi}{\times} \mathfrak{X} \right)(T) \simeq (G \times \mathfrak{X})(T)
$$

$$
\text{sending} \quad (h, \tau) \ \mapsto \ (p_1 \circ h, \tau)
$$

$$
\text{and} \quad \left((g', \sigma \circ (\iota \circ g', \pi \circ \tau')), \tau' \right) \ \leftarrow\!\shortmid \ (g', \tau'),
$$

which is functorial in T. Hence the diagram (2.2.3) is 2-Cartesian, as required. $\qquad\square$

Remark 2.2.10. To avoid confusion, we denote by $p_2 : G \times X \to X$ and $q_2 : G \times \mathfrak{X} \to \mathfrak{X}$ the second projections, respectively. Note that, the following diagram is Cartesian.

$$
\begin{array}{ccc}
G \times \mathfrak{X} & \xrightarrow{\ q_2\ } & \mathfrak{X} \\
{\scriptstyle \mathrm{Id} \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
G \times X & \xrightarrow{\ p_2\ } & X
\end{array}
$$

2.3. **G–action on the tautological invertible sheaf on the root stack.** Let $u : U \to \mathfrak{X}$ be an atlas given by the quadruple $(f_u : U \to X, M_u, \varphi_u, t_u)$; see §2.1. The tautological invertible sheaf $\mathscr{M}$ on $\mathfrak{X}$ is given by the invertible sheaf M_u on each such atlas U; see [Bor07, §3.2]. We show that the G–action on $\mathfrak{X}$, as constructed in Proposition 2.2.2, can be lifted to a G–linearization on $\mathscr{M}$.

Proposition 2.3.1. *Let U be a $\mathbb{C}$-scheme and let $u : U \to \mathfrak{X}$ be an atlas of $\mathfrak{X}$ given by the quadruple $(f_u : U \to X, M_u, \varphi_u, t_u)$. Then the Cartesian diagram of schemes*

$$
\begin{array}{ccc}
U' & \xrightarrow{\ \sigma'\ } & U \\
{\scriptstyle h_u}\downarrow & & \downarrow{\scriptstyle f_u} \\
G \times X & \xrightarrow{\ \sigma\ } & X
\end{array}
$$

induces a Cartesian diagram of the form

$$
\begin{array}{ccc}
U' & \xrightarrow{\ \sigma'\ } & U \\
{\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\
G \times \mathfrak{X} & \xrightarrow{\ \sigma_{\mathfrak{X}}\ } & \mathfrak{X}.
\end{array} \qquad\qquad (2.3.2)
$$

Proof. We first describe the left vertical map u'. Suppose $h_u : U' \to G \times X$ is given by the pair of morphisms $(\alpha_u : U' \to G, \beta_u : U' \to X)$. Consider the following composition of isomorphisms

$$(\sigma')^*(M_u^{\otimes r}) \xrightarrow[\simeq]{(\sigma')^*\varphi_u} (\sigma')^*(f_u^*(\mathcal{O}_X(D))) \qquad \qquad \beta_u^*(\mathcal{O}_X(D))$$

$$\downarrow{\simeq} \qquad\qquad\qquad\qquad\qquad\qquad \uparrow{\simeq} \qquad\qquad (2.3.3)$$

$$h_u^*(\sigma^*\mathcal{O}_X(D)) \xrightarrow[\simeq]{h_u^*\Phi^{-1}} h_u^*(p_2^*\mathcal{O}_X(D))$$

where the isomorphism at the bottom line of (2.3.3) follows from Lemma 2.2.1. Note that

$$\varphi_u(t^{\otimes r}) = f_u^*(s_D)$$
$$\implies (\sigma')^*(\varphi_u(t^{\otimes r})) = (\sigma')^*(f_u^*(s_D))$$
$$\implies ((\sigma')^*\varphi_u)((\sigma')^*t_u^{\otimes r}) = h_u^*(\sigma^*(s_D))$$
$$\implies ((\sigma')^*\varphi_u)((\sigma')^*t_u^{\otimes r}) = h_u^*(\Phi(p_2^*(s_D))), \quad \text{by Lemma 2.2.1}$$
$$\implies ((\sigma')^*\varphi_u)((\sigma')^*t_u^{\otimes r}) = h_u^*(\Phi)(h_u^*(p_2^*(s_D)))$$
$$\implies (h_u^*\Phi^{-1})\left[((\sigma')^*\varphi_u)((\sigma')^*t_u^{\otimes r})\right] = h_u^*(p_2^*(s_D)) = \beta_u^*(s_D). \qquad (2.3.4)$$

Therefore, by (2.3.3) and (2.3.4) the quadruple

$$u_0 := \left(\beta_u : U' \to X, (\sigma')^*(M_u), h_u^*(\Phi^{-1}) \circ (\sigma')^*\varphi_u, (\sigma')^*(t_u)\right)$$

defines a morphism

$$u_0 : U' \to \mathfrak{X}.$$

Now the pair of morphisms

$$(\alpha_u : U' \to G, u_0 : U' \to \mathfrak{X})$$

defines a morphism $u' : U' \to G \times \mathfrak{X}$. We show that this u' does the job. Note that, the morphism $u \circ \sigma' : U' \to \mathfrak{X}$ is given by the quadruple

$$\left(f_u \circ \sigma', (\sigma')^*(M_u), (\sigma')^*\varphi_u, (\sigma')^*t_u\right), \qquad (2.3.5)$$

and the composite morphism $\sigma_{\mathfrak{X}} \circ u' : U' \to \mathfrak{X}$ is given by the quadruple

$$\left(\sigma \circ h_u, (\sigma')^*(M_u), (h_u^*\Phi) \circ (h_u^*(\Phi^{-1}) \circ (\sigma')^*\varphi_u), (\sigma')^*t_u\right),$$

(c.f. (2.2.4)), which is equal to

$$\left(f_u \circ \sigma', (\sigma')^*(M_u), (\sigma')^*\varphi_u, (\sigma')^*t_u\right) \qquad (2.3.6)$$

since $h_u^*(\text{Id}) = \text{Id}$. Since the quadruples in (2.3.5) and (2.3.6) are the same, it follows that

$$u \circ \sigma' = \sigma_{\mathfrak{X}} \circ u',$$

i.e., the diagram (2.3.2) commutes.

To show that the diagram (2.3.2) is Cartesian, note that the objects of $(G \times \mathfrak{X}) \underset{\sigma_{\mathfrak{X}}, \mathfrak{X}, u}{\times} U$ are quadruples of the form (T, γ, x, δ), where

- T is a $\mathbb{C}$-scheme,
- $\gamma \in (G \times \mathfrak{X})(T)$,
- $x \in U(T) = \text{Hom}_{(\text{Sch}/\mathbb{C})}(T, U)$, and

- $\delta : \sigma_{\mathfrak{X}}(\gamma) \xrightarrow{\simeq} u(x)$ is an isomorphism in $\mathfrak{X}(T)$.

Let $\gamma = (\alpha, \beta) \in (G \times \mathfrak{X})(T)$, for some $\alpha : T \to G$ and $\beta : T \to \mathfrak{X}$. Suppose that β is given by the quadruple $(f_\beta : T \to X, M_\beta, \varphi_\beta, t_\beta)$, and let $h_\beta := (\alpha, f_\beta) : T \to G \times X$. Then we have

$$\sigma_{\mathfrak{X}}(\gamma) = \left(\sigma \circ h_\beta, M_\beta, h_\beta^*(\Phi) \circ \varphi_\beta, t_\beta \right); \tag{2.3.7}$$

see (2.2.4). Since $u : U \to \mathfrak{X}$ is given by the quadruple $(f_u : U \to X, M_u, \varphi_u, t_u)$, it follows that $u(x) := u \circ x$ is given by the quadruple

$$(f_u \circ x, x^*(M_u), x^*(\varphi_u), x^*(t_u)). \tag{2.3.8}$$

Since any morphism in the groupoid $\mathfrak{X}(T)$ maps to the identity map Id_T under π, it follows from (2.3.5) and (2.3.6) that $\sigma \circ h_\beta = f_u \circ x$, i.e., the following diagram commutes:

$$
\begin{array}{ccc}
T & \xrightarrow{\ x\ } & U \\
{\scriptstyle h_\beta}\big\downarrow & & \big\downarrow{\scriptstyle f_u} \\
G \times X & \xrightarrow{\ \sigma\ } & X
\end{array}
$$

Thus there exists a unique morphism $T \to U' = (G \times X) \underset{\sigma, X, u}{\times} U$. This gives a bijection

$$\left((G \times \mathfrak{X}) \underset{\sigma_{\mathfrak{X}}, \mathfrak{X}, u}{\times} U \right)(T) \simeq U'(T)$$

which is functorial in T, proving that the diagram (2.3.2) is Cartesian.   □

Corollary 2.3.9. *For any atlas $u : U \to \mathfrak{X}$ of $\mathfrak{X}$ with U a $\mathbb{C}$-scheme, its pullback $u' : U' \to G \times \mathfrak{X}$, as defined in Proposition 2.3.1, is an atlas of $G \times \mathfrak{X}$.*

Proof. To show that u' gives an atlas, we need to check that for any scheme $t : T \to G \times \mathfrak{X}$, the projection map $T \underset{t, G \times \mathfrak{X}, u'}{\times} U' \to T$ is smooth and surjective [Ols16, Definition 8.1.4]. It follows from Proposition 2.3.1 that

$$
\begin{aligned}
T \underset{t, G \times \mathfrak{X}, u'}{\times} U' &\simeq T \underset{t, G \times \mathfrak{X}, u'}{\times} \left((G \times \mathfrak{X}) \underset{\sigma_{\mathfrak{X}}, \mathfrak{X}, u}{\times} U \right) \\
&\simeq T \underset{(\sigma_{\mathfrak{X}} \circ t), \mathfrak{X}, u}{\times} U.
\end{aligned}
$$

Since $U \xrightarrow{u} \mathfrak{X}$ is an atlas, the morphism $T \underset{(\sigma_{\mathfrak{X}} \circ t), \mathfrak{X}, u}{\times} U \to T$ is a smooth surjection. Hence the result follows.   □

Lemma 2.3.10. *With the above notations, $\sigma_{\mathfrak{X}} : G \times \mathfrak{X} \to \mathfrak{X}$ is a smooth morphism of algebraic stacks.*

Proof. We need to check that for some atlases $v : V \to G \times \mathfrak{X}$ and $u : U \to \mathfrak{X}$, there exists a commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\ f\ } & U \\
{\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle u} \\
G \times \mathfrak{X} & \xrightarrow{\ \sigma_{\mathfrak{X}}\ } & \mathfrak{X}
\end{array}
$$

such that f is smooth (see [Hei10, Definition 2.4]). Choose an atlas $u : U \to \mathfrak{X}$ given by the quadruple $(f_u, M_u, \varphi_u, t_u)$, and consider the following Cartesian diagram

$$
\begin{array}{ccc}
U' & \xrightarrow{\sigma'} & U \\
{\scriptstyle h_u}\downarrow & & \downarrow{\scriptstyle f_u} \\
G \times X & \xrightarrow{\sigma} & X.
\end{array}
$$

Since the action morphism $\sigma : G \times X \to X$ is smooth, so is its pullback $\sigma' : U' \to U$. It follows from Corollary 2.3.9 that the pullback $u' : U' \to G \times \mathfrak{X}$ of u is an atlas on $G \times \mathfrak{X}$. Thus, we have a commutative diagram

$$
\begin{array}{ccc}
U' & \xrightarrow{\sigma'} & U \\
{\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\
G \times \mathfrak{X} & \xrightarrow{\sigma_{\mathfrak{X}}} & \mathfrak{X}
\end{array}
$$

where u and u' are atlases and σ' is smooth. Thus $\sigma_{\mathfrak{X}}$ is a smooth morphism of stacks. $\qquad\square$

Since $\sigma_{\mathfrak{X}} : G \times \mathfrak{X} \to \mathfrak{X}$ is a smooth morphism, it induces a morphism of Lisse-étale sites

$$
\text{Lis-Ét}(G \times \mathfrak{X}) \longrightarrow \text{Lis-Ét}(\mathfrak{X})
$$

(see [Sta24, Tag 0GR1] and [Ols07]), which enables us to define pull-back $\sigma_{\mathfrak{X}}^*(\mathcal{F})$ of a coherent sheaf $\mathcal{F} \in \mathfrak{X}_{\text{lis-ét}}$ as follows. Given a smooth morphism $t : T \to G \times \mathfrak{X}$, the composition $\sigma_{\mathfrak{X}} \circ t : T \to \mathfrak{X}$ is smooth by Lemma 2.3.10. Following notation and proof of [Ols07, Lemma 6.5], we define

$$
(\sigma_{\mathfrak{X}}^* \mathcal{F})_t := \mathcal{F}_{\sigma_{\mathfrak{X}} \circ t} .
$$

Proposition 2.3.11. *The tautological invertible sheaf $\mathcal{M}$ on $\mathfrak{X}$ admits a linearization of the G-action on $\mathfrak{X}$. Consequently, every tensor power of $\mathcal{M}$ on $\mathfrak{X}$ also admits a linearization.*

Proof. Let V be a $\mathbb{C}$-scheme and let $v : V \to G \times \mathfrak{X}$ be a smooth morphism given by a pair of morphisms $\alpha_v : V \to G$ and $\beta_v : V \to \mathfrak{X}$, where β_v is given by the quadruple

$$
(f_v : V \to X, M_v, \varphi_v, t_v). \tag{2.3.12}
$$

Let $h_v := (\alpha_v, f_v) : V \to G \times X$. Then $\sigma_{\mathfrak{X}} \circ v : V \to \mathfrak{X}$ is given by the quadruple

$$
\sigma_{\mathfrak{X}}(\alpha_v, \beta_v) = \left(\sigma \circ (\alpha_v, f_v) : V \to X, M_v, \left((\alpha_v, f_v)^* \Phi \right) \circ \varphi_v, t_v \right),
$$

(see (2.2.4)), which implies that

$$
(\sigma_{\mathfrak{X}}^* \mathcal{M})_v = \mathcal{M}_{\sigma_{\mathfrak{X}} \circ v} = M_v.
$$

Similarly, for the second projection $q_2 : G \times \mathfrak{X} \to \mathfrak{X}$, the map $q_2 \circ v = \beta_v : V \to \mathfrak{X}$ is given by the quadruple (2.3.12), and thus

$$
(q_2^* \mathcal{M})_v = \mathcal{M}_{q_2 \circ v} = \mathcal{M}_{\beta_v} = M_v.
$$

Thus there exists a natural isomorphism

$$
\sigma_{\mathfrak{X}}^* \mathcal{M} \simeq q_2^* \mathcal{M} .
$$

The second statement about tensor powers is obvious. $\qquad\square$

3. BISWAS–BORNE CORRESPONDENCE FOR EQUIVARIANT BUNDLES

In [Bis97] it is shown that the category of parabolic bundles on X with parabolic structure defined along a simple normal crossing divisor D with rational parabolic weights is equivalent to the category of orbifold vector bundles on certain finite Galois cover of X. The notion of parabolic bundles is reformulated in the language of functors in [Yok95], and the main result of [Bis97] is reformulated and generalized using the language of root stacks by N. Borne in [Bor07]. We begin with the definition of parabolic bundles from [Yok95].

3.1. **Parabolic bundles.**

Let $\mathbb{R}$ be the category whose objects are real numbers and given $r, s \in \mathbb{R}$ we define
$$\mathrm{Mor}_{\mathbb{R}}(r,s) = \begin{cases} \{\iota^{r,s}\}, & \text{if} \quad r \leq s, \\ \varnothing, & \text{if} \quad r > s. \end{cases}$$
Let $\mathbb{R}^{\mathrm{op}}$ be the opposite category of $\mathbb{R}$. An $\mathbb{R}$-*filtered $\mathcal{O}_X$-module* is a functor $E_\bullet : \mathbb{R}^{\mathrm{op}} \to \mathfrak{Mod}(\mathcal{O}_X)$, where $\mathfrak{Mod}(\mathcal{O}_X)$ is the category of $\mathcal{O}_X$-modules on X. We denote by E_t the $\mathcal{O}_X$-module $E_\bullet(t)$, for all $t \in \mathbb{R}$. For $s \geq t$ in $\mathbb{R}$, we denote by $\iota_{E_\bullet}^{s,t}$ the $\mathcal{O}_X$-module homomorphism
$$\iota_{E_\bullet}^{s,t} := E_\bullet(\iota^{s,t}) : E_s \to E_t.$$

Morphisms of $\mathbb{R}$-filtered $\mathcal{O}_X$-modules are given by natural transformations of functors. Let $\mathbf{Filt}_{\mathbb{R}}(\mathfrak{Mod}(\mathcal{O}_X))$ be the category whose objects are $\mathbb{R}$–filtered $\mathcal{O}_X$-modules on X and morphisms are as defined above. Given $E_\bullet \in \mathbf{Filt}_{\mathbb{R}}(\mathfrak{Mod}(\mathcal{O}_X))$ and $s \in \mathbb{R}$, we define
$$E_\bullet[s] : \mathbb{R}^{\mathrm{op}} \to \mathfrak{Mod}(\mathcal{O}_X)$$
by setting
$$E_\bullet[s](t) := E_\bullet(s+t), \quad \text{and} \quad E_\bullet[s](\iota^{t,t'}) = \iota_{E_\bullet}^{s+t,s+t'},$$
for all $t, t' \in \mathbb{R}$. For $s \geq t$ in $\mathbb{R}$, we have a morphism of functors
$$\iota_{E_\bullet}^{[s,t]} : E_\bullet[s] \to E_\bullet[t]$$
given by the unique morphism
$$\iota_{E_\bullet}^{s+u,t+u} : E_{s+u} \to E_{t+u}, \ \forall \, u \in \mathbb{R}.$$

Given $E_\bullet \in \mathbf{Filt}_{\mathbb{R}}(\mathfrak{Mod}(\mathcal{O}_X))$ and an $\mathcal{O}_X$-module V on X, we have an $\mathbb{R}$-filtered $\mathcal{O}_X$-module $E_\bullet \otimes F$ defined by $(E_\bullet \otimes F)(t) := E_t \otimes F$, $\forall \, t \in \mathbb{R}$. Similarly, tensoring $E_\bullet$ with an $\mathcal{O}_X$-module homomorphism $f : F \to G$, we get we get a morphism $\mathrm{Id}_{E_\bullet} \otimes f : E_\bullet \otimes F \to E_\bullet \otimes G$ in $\mathbf{Filt}_{\mathbb{R}}(\mathfrak{Mod}(\mathcal{O}_X))$.

Fix an integer $r \geq 1$, and let $\frac{1}{r}\mathbb{Z}$ be the full subcategory of $\mathbb{R}$ whose objects are elements of the set $\frac{1}{r}\mathbb{Z}$. We denote by $\mathbf{Vect}(X)$ the category of *vector bundles* (locally free coherent sheaves of finite ranks) on X. Following above notations, the notion of $\frac{1}{r}\mathbb{Z}$–*filtered vector bundles* on X makes sense.

Definition 3.1.1. A *parabolic vector bundle* on X with a parabolic structure along D with rational parabolic weights in $\frac{1}{r}\mathbb{Z}$ is a pair $(E_\bullet, j_{E_\bullet})$, where
$$E_\bullet : \left(\frac{1}{r}\mathbb{Z}\right)^{\mathrm{op}} \longrightarrow \mathbf{Vect}(X)$$

is an $\frac{1}{r}\mathbb{Z}$–filtered vector bundle on X and

$$j_{E_\bullet} : E_\bullet \otimes \mathcal{O}_X(-D) \longrightarrow E_\bullet[1]$$

is an isomorphism of functors such that the following diagram commutes.

$$
\begin{array}{ccc}
E_\bullet \otimes \mathcal{O}_X(-D) & \xrightarrow[\simeq]{\ j_{E_\bullet}\ } & E_\bullet[1] \\
& \searrow{\scriptstyle \mathrm{Id}_{E_\bullet} \otimes \iota_D} \quad \swarrow{\scriptstyle \iota_{E_\bullet}^{[1,0]}} & \\
& E_\bullet &
\end{array}
\qquad (3.1.2)
$$

where $\iota_D : \mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$ is the natural inclusion map of $\mathcal{O}_X$-modules. We simply call $E_\bullet$ a parabolic bundle, when the data of D and r is clear from the context. A morphism of parabolic bundles from $E_\bullet$ to $F_\bullet$ is a natural transformation of functors compatible with the diagram (3.1.2) in the obvious sense. We denote by $\mathbf{ParVect}(X, r, D)$ the category of parabolic vector bundles on X having parabolic structure along D and rational parabolic weights in $\frac{1}{r}\mathbb{Z}$.

Then we have the following result due to N. Borne.

Theorem 3.1.3. [Bor07, Théorème 3.13] *With the above notations, there is an equivalence of categories between* $\mathbf{Vect}(\mathfrak{X})$ *and* $\mathbf{ParVect}(X, r, D)$ *given by the following functors:*

- *A vector bundle $\mathcal{E}$ on $\mathfrak{X}$ gives rise to the parabolic vector bundle $E_\bullet$ on X defined by the functor*

$$\frac{\ell}{r} \longmapsto E_{\frac{\ell}{r}} := \pi_* \left(\mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathscr{M}^{-\ell} \right), \ \forall \ell \in \mathbb{Z}.$$

- *Conversely, a parabolic vector bundle $E_\bullet$ on X gives rise to the vector bundle*

$$\mathcal{E} := \int^{\frac{1}{r}\mathbb{Z}} \pi^*(E_\bullet) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathscr{M}^{\bullet r}$$

on the root stack $\mathfrak{X}$, where $\mathscr{M}$ is the tautological invertible sheaf on $\mathfrak{X}$ satisfying $\pi^*\mathcal{O}_X(D) \cong \mathscr{M}^r$, and $\int^{\frac{1}{r}\mathbb{Z}}$ stands for the coend.

3.2. **G–equivariance structure.** Let G be an affine algebraic group over $\mathbb{C}$. We assume that G has no non-trivial characters (c.f. Proposition 2.2.2). In this subsection we define the notion of G–equivariant parabolic vector bundles on X, and the notion of G–equivariant vector bundles on the root stack $\mathfrak{X}$, and then formulate Biswas–Borne correspondence (Theorem 3.1.3) in G–equivariant setup. Let

$$E_\bullet : \left(\frac{1}{r}\mathbb{Z} \right)^{\mathrm{op}} \to \mathbf{Vect}(X)$$

be a parabolic vector bundle on X. If $f : Y \to X$ is a smooth morphism of $\mathbb{C}$-schemes, then f^*D is a simple normal crossing divisor on Y (see [Sta24, Section 0CBN]) and the functor

$$f^*(E_\bullet) : \left(\frac{1}{r}\mathbb{Z} \right)^{\mathrm{op}} \longrightarrow \mathbf{Vect}(X), \ \frac{\ell}{r} \mapsto f^*(E_{\frac{\ell}{r}}), \qquad (3.2.1)$$

defines a parabolic vector bundle on Y with rational parabolic weights in $\frac{1}{r}\mathbb{Z}$ and having parabolic structure along f^*D, called the *pullback of $E_\bullet$ along f*. Since the G–action map σ and the second projection map p_2 from $G \times X$ to X are smooth and $p_2^*(D) = \sigma^*D$ by assumption, both

$\sigma^*(E_\bullet)$ and $p_2^*(E_\bullet)$ are parabolic bundles on $G \times X$ with rational parabolic weights in $\frac{1}{r}\mathbb{Z}$ and having parabolic structures along $\sigma^*D = p_2^*D$.

Definition 3.2.2. With the above notations, a *G-equivariance structure* on a parabolic vector bundle $E_\bullet$ is a natural isomorphism of functors

$$\psi_\bullet : p_2^*(E_\bullet) \longrightarrow \sigma^*(E_\bullet),$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
p_2^*(E_\bullet \otimes \mathcal{O}_X(-D)) & \xrightarrow{\psi_\bullet \otimes \Phi_{-1}} & \sigma^*(E_\bullet \otimes \mathcal{O}_X(-D)) \\
{\scriptstyle p_2^*(j_{E_\bullet})} \downarrow & & \downarrow {\scriptstyle \sigma^*(j_{E_\bullet[1]})} \\
p_2^*(E_\bullet[1]) & \xrightarrow{\psi_\bullet[1]} & \sigma^*(E_\bullet[1])
\end{array}
$$

where Φ_{-1} is the isomorphism induced from Lemma 2.2.1. A *G-equivariant parabolic vector bundle* is a pair $(E_\bullet, \psi_\bullet)$, where $E_\bullet$ is a parabolic vector bundle and $\psi_\bullet$ is a G-equivariance structure on $E_\bullet$.

   Let $(E_\bullet, \phi_\bullet)$ and $(F_\bullet, \psi_\bullet)$ be two G-equivariant parabolic vector bundles on X. A morphism from $(E_\bullet, \phi_\bullet)$ to $(F_\bullet, \psi_\bullet)$ is a morphism of parabolic bundles

$$f_\bullet : E_\bullet \to F_\bullet$$

such that the following diagram is 2-commutative.

$$
\begin{array}{ccc}
p_2^*(E_\bullet) & \xrightarrow{p_2^*(f_\bullet)} & p_2^*(F_\bullet) \\
{\scriptstyle \phi_\bullet} \downarrow & & \downarrow {\scriptstyle \psi_\bullet} \\
\sigma^*(E_\bullet) & \xrightarrow{\sigma^*(f_\bullet)} & \sigma^*(F_\bullet)
\end{array}
$$

We denote by **ParVect**$^G(X, r, D)$ the category of G–equivariant parabolic vector bundles on X with rational parabolic weights in $\frac{1}{r}\mathbb{Z}$ supported on D.

   Let $\mathfrak{Coh}(\mathfrak{X})$ be the category of coherent sheaves of $\mathcal{O}_{\mathfrak{X}}$–modules on $\mathfrak{X}$. Let $\sigma_{\mathfrak{X}} : G \times \mathfrak{X} \to \mathfrak{X}$ be the G–action on $\mathfrak{X}$ (c.f. Proposition 2.2.2), and let $q_2 : G \times \mathfrak{X} \to \mathfrak{X}$ be the projection morphism onto the second factor.

Definition 3.2.3. (a) A *G–linearization* on a coherent sheaf $\mathcal{E}$ on $\mathfrak{X}$ is an isomorphism

$$\psi : q_2^* \mathcal{E} \xrightarrow{\simeq} \sigma_{\mathfrak{X}}^* \mathcal{E}$$

of sheaves of $\mathcal{O}_{G \times \mathfrak{X}}$–modules such that the following triangle is 2–commutative:

$$
\begin{array}{ccc}
q_{23}^* q_2^* \mathcal{E} & \xrightarrow{q_{23}^* \psi} & q_{23}^* \sigma^* \mathcal{E} \\
\| & & \| \\
(\mu \times \mathrm{Id}_{\mathfrak{X}})^* p_2^* \mathcal{E} & & (\mathrm{Id}_G \times \sigma_{\mathfrak{X}})^* q_2^* \mathcal{E} \\
{\scriptstyle (\mu \times \mathrm{Id}_{\mathfrak{X}})^* \psi} \downarrow & & \downarrow {\scriptstyle (\mathrm{Id}_G \times \sigma_{\mathfrak{X}})^* \psi} \\
(\mu \times \mathrm{Id}_{\mathfrak{X}})^* \sigma_{\mathfrak{X}}^* \mathcal{E} & == & (\mathrm{Id}_G \times \sigma_{\mathfrak{X}})^* \sigma_{\mathfrak{X}}^* \mathcal{E},
\end{array}
$$

where $\mu : G \times G \to G$ is the group operation on G and $q_{23} : G \times G \times \mathfrak{X} \to G \times \mathfrak{X}$ is the projection morphism onto the second and third factors.

(b) A *G-equivariant coherent sheaf* on $\mathfrak{X}$ is a pair $(\mathcal{E}, \psi)$, where $\mathcal{E}$ is a coherent sheaf on $\mathfrak{X}$ and Ψ is a G–linearization on $\mathcal{E}$.

(c) A *morphism* $(\mathcal{E}, \psi) \to (\mathcal{E}', \psi')$ of G–equivariant coherent sheaves on $\mathfrak{X}$ is an $\mathcal{O}_{\mathfrak{X}}$–module homomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that the following diagram is 2–commutative.

$$
\begin{array}{ccc}
q_2^* \mathcal{E} & \xrightarrow{\ \ \psi\ \ } & \sigma_{\mathfrak{X}}^* \mathcal{E} \\
{\scriptstyle q_2^*(f)} \downarrow & & \downarrow {\scriptstyle \sigma_{\mathfrak{X}}^*(f)} \\
q_2^* \mathcal{E}' & \xrightarrow{\ \ \psi'\ \ } & \sigma_{\mathfrak{X}}^* \mathcal{E}'
\end{array}
$$

Let $\mathfrak{Coh}^G(\mathfrak{X})$ be the category of G–equivariant coherent sheaves on $\mathfrak{X}$, and let $\mathbf{Vect}^G(\mathfrak{X})$ be the full subcategory of $\mathfrak{Coh}^G(\mathfrak{X})$ whose objects are G–equivariant vector bundle on $\mathfrak{X}$.

Theorem 3.2.4. *With the above assumptions, there is an equivalence of categories between* $\mathbf{Vect}^G(\mathfrak{X})$ *and* $\mathbf{ParVect}^G(X, r, D)$.

Proof. It suffices to show that the equivalence in Theorem 3.1.3 preserves the G-equivariance structures. Let $(E_\bullet, \psi_\bullet) \in \mathbf{ParVect}^G(X, r, D)$ be a G–equivariant parabolic vector bundle on X, and let

$$
\mathcal{E} := \int^{\frac{1}{r}\mathbb{Z}} \pi^*(E_\bullet) \otimes \mathcal{M}^{\bullet r}
$$

be the associated vector bundle on the root stack $\mathfrak{X}$ obtained from $E_\bullet$ via the coend (certain colimit) construction in the category $\mathbf{Vect}(\mathfrak{X})$ of vector bundles on $\mathfrak{X}$ [Bor07]. Since q_2^* is left adjoint to q_{2_*} (see [Ols16, Proposition 9.3.6]), it follows that the vector bundle $q_2^* \mathcal{E} \in \mathbf{Vect}(G \times \mathfrak{X})$ can be interpreted as the following coend:

$$
q_2^* \mathcal{E} = \int^{\frac{1}{r}\mathbb{Z}} q_2^* \left(\pi^*(E_\bullet) \otimes \mathcal{M}^{\bullet r} \right).
$$

Similarly the vector bundle $\sigma_{\mathfrak{X}}^* \mathcal{E} \in \mathbf{Vect}(G \times \mathfrak{X})$ can be interpreted as the following coend:

$$
\sigma_{\mathfrak{X}}^* \mathcal{E} = \int^{\frac{1}{r}\mathbb{Z}} \sigma_{\mathfrak{X}}^* \left(\pi^*(E_\bullet) \otimes \mathcal{M}^{\bullet r} \right).
$$

Now, consider the natural isomorphisms of bifunctors

$$
\left(\frac{1}{r}\mathbb{Z} \right)^{\mathrm{op}} \times \frac{1}{r}\mathbb{Z} \longrightarrow \mathbf{Vect}(\mathfrak{X})
$$

given by the following sequence of isomorphisms:

$$
\begin{aligned}
q_2^* \left(\pi^*(E_\bullet) \otimes \mathcal{M}^{\bullet r} \right) &\simeq q_2^*(\pi^*(E_\bullet)) \otimes q_2^* \mathcal{M}^{\bullet r} \\
&\simeq (\mathrm{Id} \times \pi)^*(p_2^*(E_\bullet)) \otimes p_2^*(\mathcal{M}^{\bullet r}) \quad \text{[see Remark 2.2.10]} \\
&\simeq (\mathrm{Id} \times \pi)^*(\sigma_{\mathfrak{X}}^*(E_\bullet)) \otimes \sigma_{\mathfrak{X}}^*(\mathcal{M}^{\bullet r}) \quad \text{[Proposition 2.3.11]} \\
&\simeq \sigma_{\mathfrak{X}}^*(\pi^*(E_\bullet)) \otimes \sigma_{\mathfrak{X}}^*(\mathcal{M}^{\bullet r}) \qquad \text{[from diagram (2.2.3)]} \\
&\simeq \sigma_{\mathfrak{X}}^*(\pi^*(E_\bullet) \otimes \mathcal{M}^{\bullet r}).
\end{aligned}
$$

Then it follows from Remark 3.2.6 (see below) that the coend of $q_2^*\left(\pi^*(E_\bullet) \otimes \mathscr{M}^{\bullet\,r}\right)$ and the coend of $\sigma_{\mathfrak{X}}^*\left(\pi^*(E_\bullet) \otimes \mathscr{M}^{\bullet\,r}\right)$ are naturally isomorphic. Thus, we have a natural isomorphism

$$\varphi : q_2^*(\mathcal{E}) \xrightarrow{\simeq} \sigma_{\mathfrak{X}}^*(\mathcal{E}) \tag{3.2.5}$$

of vector bundles over $G \times \mathfrak{X}$ which gives the required G–equivariance structure on $\mathcal{E}$ (see Definition 3.2.3).

Conversely, let $\mathcal{F} \in \mathbf{Vect}^G(\mathfrak{X})$ be a G–equivaraint vector bundle on $\mathfrak{X}$. Consider the following Cartesian diagram (2.2.3) from Proposition 2.2.2, namely

$$
\begin{array}{ccc}
G \times \mathfrak{X} & \xrightarrow{\ \sigma_{\mathfrak{X}}\ } & \mathfrak{X} \\
{\scriptstyle \mathrm{Id}_G \times \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
G \times X & \xrightarrow{\ \sigma\ } & X
\end{array}
$$

Since π is a finite morphism [Bor07, Corrolaire 3.6] and σ is smooth (in particular flat), we have an isomorphism of functors $\sigma^* \circ \pi_* \simeq (\mathrm{Id} \times \pi)_* \circ \sigma_{\mathfrak{X}}^*$, and similarly, $p_2^* \circ \pi_* \simeq (\mathrm{Id} \times \pi)_* \circ q_2^*$ (see Remark 2.2.10). It follows that for all $\ell \in \mathbb{Z}$,

$$
\begin{aligned}
p_2^*(\pi_*(\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathscr{M}^{-\ell})) &\simeq (\mathrm{Id} \times \pi)_* \left(q_2^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathscr{M}^{-\ell}) \right) \\
&\simeq (\mathrm{Id} \times \pi)_* \left(\sigma_{\mathfrak{X}}^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathscr{M}^{-\ell}) \right), \quad [\text{see Proposition } 2.3.11] \\
&\simeq \sigma^* \left(\pi_*(\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathscr{M}^{-\ell}) \right).
\end{aligned}
$$

In other words, if $F_\bullet$ is the parabolic bundle corresponding to $\mathcal{F}$ in Theorem 3.1.3, we obtain isomorphisms

$$\phi_{\frac{\ell}{r}} : p_2^*(F_{\frac{\ell}{r}}) \xrightarrow{\simeq} \sigma^*(F_{\frac{\ell}{r}}), \ \ \forall\, \ell \in \mathbb{Z},$$

which is functorial in $\ell \in \mathbb{Z}$. It is easy to see that the diagrams in Definition 3.2.2 under these $\phi_{\frac{\ell}{r}}$'s also commute. Thus $F_\bullet$ has an induced G–equivariance structure. $\qquad\square$

Remark 3.2.6. Let $\mathscr{C}$ be any category, and let $F, G : J \to \mathscr{C}$ be two functors from an index category J such that both the colimits $\varinjlim_{j \in J} F_j$ and $\varinjlim_{j \in J} G_j$ exist in $\mathcal{C}$. Then a natural isomorphism of functors $\varphi := \{\varphi_j\}_{j \in J} : F \to G$ gives rise to a unique isomorphism

$$\widetilde{\varphi} : \varinjlim_{j \in J} F_j \longrightarrow \varinjlim_{j \in J} G_j$$

commuting the following diagrams for all $j \in J$.

$$
\begin{array}{ccc}
F_j & \longrightarrow & \varinjlim\limits_{j \in J} F_j \\
{\scriptstyle \varphi_j}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\varphi}} \\
G_j & \longrightarrow & \varinjlim\limits_{j \in J} G_j
\end{array}
$$

This being a straightforward application of the universal property of colimits, we omit its proof.

4. BISWAS-BORNE CORRESPONDENCE FOR EQUIVARIANT CONNECTIONS

In this section, we assume that D is a smooth irreducible Cartier divisor on X. We now describe the notion of logarithmic connections on (X, D). We denote by $\Omega^1_{X/\mathbb{C}}(\log D)$ the sheaf of meromorphic differentials on X having at most logarithmic poles along D.

4.1. Parabolic connections.
Let E be a vector bundle on X.

Definition 4.1.1. (i) A *logarithmic connection* on E is a $\mathbb{C}$-linear sheaf homomorphism

$$\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}(\log D) \qquad (4.1.2)$$

satisfying the Leibniz rule:

$$\nabla(f \cdot s) = f\nabla s + s \otimes df,$$

for all locally defined sections f and s of $\mathcal{O}_X$ and E, respectively. It is usually denoted as a pair (E, ∇). If the image of ∇ lands in the subsheaf $E \otimes \Omega^1_{X/\mathbb{C}}$, then ∇ is said to be a *holomorphic connection* on E.

(ii) A *morphism of logarithmic connections* $(E, \nabla) \to (E', \nabla')$ is given by a vector bundle morphism $\phi : E \longrightarrow E'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
E & \xrightarrow{\ \nabla\ } & E \otimes \Omega^1_{X/\mathbb{C}}(\log D) \\
\phi \downarrow & & \downarrow \phi \otimes \mathrm{Id} \\
E' & \xrightarrow{\ \nabla'\ } & E' \otimes \Omega^1_{X/\mathbb{C}}(\log D)\,.
\end{array}
$$

Let $\mathrm{Con}(X, D)$ be the category of logarithmic connections on (X, D).

Definition 4.1.3. A *parabolic connection* on (X, D) having rational parabolic weights in $\frac{1}{r}\mathbb{Z}$ is a functor

$$(E_\bullet, \nabla_\bullet) : \left(\frac{1}{r}\mathbb{Z}\right)^{\mathrm{op}} \longrightarrow \mathrm{Con}(X, D)$$

such that for each $n \in \mathbb{N}$ there is a natural isomorphism of functors

$$\left(E_\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_X(-nD)\,,\ \nabla_\bullet(-nD)\right) \xrightarrow{\sim} \left(E_{\bullet+n}\,,\ \nabla_{\bullet+n}\right)$$

such that the following diagram commutes

$$
\begin{array}{ccc}
(E_{\bullet+n}\,,\ \nabla_{\bullet+n}) & \xrightarrow{\hspace{3cm}} & (E_\bullet\,,\ \nabla_\bullet) \\
\simeq \uparrow & \nearrow_{\ \mathrm{Id}_{E_\bullet} \otimes \iota_n} & \\
(E_\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_X(-nD)\,,\ \nabla_\bullet(-nD)) & &
\end{array}
\qquad (4.1.4)
$$

where ι_n stands for the canonical inclusion $\mathcal{O}(-nD) \subset \mathcal{O}_X$.

4.2. **Equivariant parabolic connection.** Let $\sigma : G \times X \to X$ be an action of a connected affine algebraic group G on X and satisfying $\sigma^*(D) = p_2^*(D)$. For each $g \in G$, let

$$\sigma_g : X \longrightarrow X$$

be the map given by multiplication by g. Consider the sheaf $\Omega^1_{X/\mathbb{C}}(\log D)$ on X, which is locally free [EV92, Properties 2.2]. Clearly, the assumption $\sigma^*(D) = p_2^*(D)$ gives rise to a linearization on $\Omega^1_{X/\mathbb{C}}(\log D)$:

$$\psi : p_2^*\left(\Omega^1_{X/\mathbb{C}}(\log D)\right) \xrightarrow{\simeq} \sigma^*\left(\Omega^1_{X/\mathbb{C}}(\log D)\right).$$

Thus, for each $g \in G$, we get an isomorphism

$$\psi_g : \Omega^1_{X/\mathbb{C}}(\log D) \xrightarrow{\simeq} \sigma_g^*\left(\Omega^1_{X/\mathbb{C}}(\log D)\right)$$

of locally free sheaves. Moreover, the canonical sheaf morphism

$$\sigma_g^*\left(\Omega^1_{X/\mathbb{C}}(\log D)\right) \longrightarrow \Omega^1_{X/\mathbb{C}}(\log \sigma_g^*(D))$$

is an isomorphism for each $g \in G$ [BL23, Lemma 3.5]. Composing these isomorphisms, we get the following collection of isomorphisms of locally free sheaves:

$$\delta_g : \Omega^1_{X/\mathbb{C}}(\log D) \xrightarrow{\simeq} \Omega^1_{X/\mathbb{C}}(\log \sigma_g^*(D)), \ \forall \, g \in G. \tag{4.2.1}$$

Following [BH13] we define the notion of G-equivaraint logarithmic connection and G-equivariant parabolic connection as follows.

Definition 4.2.2.   (i) A *G-equivariant logarithmic connection on (X, D)* is given by a logarithmic connection $(E, \widehat{\nabla})$ on (X, D) together with a collection of isomorphisms

$$\phi_g : E \xrightarrow{\simeq} \sigma_g^*(E), \ \forall \, g \in G,$$

making the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc}
E & \xrightarrow{\ \widehat{\nabla}\ } & E \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}(\log D) \\
{\scriptstyle \phi_g}\downarrow & & \downarrow{\scriptstyle \phi_g \otimes \delta_g} \\
\sigma_g^*(E) & \xrightarrow{\ \sigma_g^*(\widehat{\nabla})\ } & \sigma_g^*(E) \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}(\log \sigma_g^*(D)),
\end{array} \tag{4.2.3}$$

where δ_g is the isomorphism in (4.2.1).

 (ii) A *G-equivariant parabolic connection on (X, D)* is given by a parabolic connection $(E_\bullet, \widehat{\nabla}_\bullet)$ on (X, D) (see Definition 4.1.3) equipped with a collection of natural isomorphisms

$$\phi_{h,\bullet} : E_\bullet \simeq \sigma_h^*(E_\bullet) \ \ \forall \, h \in G \tag{4.2.4}$$

satisfying $\sigma_g^*(\phi_{h,\bullet}) \circ \phi_{g,\bullet} = \phi_{hg,\bullet}$ for all $g, h \in G$, where we canonically identify $\sigma_g^*(\sigma_h^*(E_\bullet)) \simeq \sigma_{hg}^*(E_\bullet)$. These natural isomorphisms are also required to make the following diagrams commute for all $h \in G$:

$$\begin{array}{ccc}
E_\bullet & \xrightarrow{\ \widehat{\nabla}_\bullet\ } & E_\bullet \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}(\log D) \\
{\scriptstyle \phi_{h,\bullet}}\downarrow & & \downarrow{\scriptstyle \phi_{h,\bullet} \otimes \delta_h} \\
\sigma_h^*(E_\bullet) & \xrightarrow{\ \sigma_h^*(\widehat{\nabla}_\bullet)\ } & \sigma_h^*(E_\bullet) \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}(\log \sigma_h^*(D)),
\end{array} \tag{4.2.5}$$

where δ_h is the isomorphism as in (4.2.1).

The following result shows existence of a canonical logarithmic connection on the line bundle $\mathcal{O}_X(D)$ which we use later (c.f. [EV92]).

Lemma 4.2.6. [BL23, Lemma 3.7] *Let $B = \sum_{i \in I} \mu_i D_i$ be a Cartier divisor with support in D. Then there exists a canonical logarithmic connection:*

$$d(B) : \mathcal{O}_X(B) \to \mathcal{O}_X(B) \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}(\log D) \tag{4.2.7}$$

characterized by the equation $d(B) \left(\prod_{i \in I} x_i^{-\mu_i} \right) = - \sum_{i \in I} x_i^{-\mu_i} \cdot \sum_{i \in I} \mu_i \dfrac{dx_i}{x_i}$, where x_i is a local equation of D_i.

Lemma 4.2.8. *With the assumptions as in the beginning of this section, the pair $(\mathcal{O}_X(D), d(D))$ is a G-equivariant logarithmic connection on (X, D) in the sense of Definition 4.2.2.*

Proof. The conditions $\sigma_g^*(D) = D$ give rise to a collection of isomorphims $\theta_g : \mathcal{O}(D) \xrightarrow{\simeq} \sigma_g^* \mathcal{O}_X(D)$ of line bundles. These conditions imply that the local equation of D differ by multiplying by a unit after composing with σ_g. From this, it is easy to see that the diagram (4.2.3) for $(\mathcal{O}_X(D), d(D))$ commutes by taking $\phi_g = \theta_g$ and $\widehat{\nabla} = d(D)$ in Definition 4.2.2. □

Remark 4.2.9. There is a well-defined notion of tensor product of two logarithmic connections, which is again a logarithmic connection [BL23, 3.3.2]. Thus, given a vector bundle E admitting a logarithmic connection ∇ with poles along D, and a Cartier divisor B with support in D, one can *twist* the connection ∇ by B by considering the tensor product connection on $E \otimes \mathcal{O}_X(B)$. We shall denote this twisted connection by $\nabla(B)$. The tensor product of two G-equivariant logarithmic connections is again G-equivariant.

4.3. **Equivariant logarithmic connection on root stack.** The tautological invertible sheaf $\mathscr{M}$ on the root stack $\mathfrak{X}$ gives rise to a smooth irreducible effective Cartier divisor $\mathfrak{D}$ on $\mathfrak{X}$ satisfying $\pi^*(D) = r\mathfrak{D}$ (see [BL23, Remark 2.7]). Note that $\mathscr{M} \simeq \mathcal{O}_{\mathfrak{X}}(\mathfrak{D})$. As we have seen in Proposition 2.2.2, under the additional assumption that G has no non-trivial character, the G-action can be lifted to a G–action $\sigma_{\mathfrak{X}} : G \times \mathfrak{X} \longrightarrow \mathfrak{X}$ such that the natural map $\pi : \mathfrak{X} \longrightarrow X$ is G-equivariant. The notion of logarithmic connections with poles along a strict normal crossings divisor makes sense for smooth Deligne-Mumford stacks as well. Thus, we can speak about equivariant logarithmic connections on the root stack $\mathfrak{X}$. Now, as in the case of schemes, the linearization on $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})$ gives rise to the following isomorphisms on the sheaf of logarithmic differentials on the root stack:

$$\eta_g : \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \mathfrak{D}) \xrightarrow{\simeq} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \sigma_{\mathfrak{X},g}^*(\mathfrak{D})), \ \forall \, g \in G. \tag{4.3.1}$$

Definition 4.3.2. A *G-equivariant logarithmic connection on* $(\mathfrak{X}, \mathfrak{D})$ is a logarithmic connection

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \mathfrak{D})$$

such that $\mathcal{E}$ is a G-equivariant vector bundle on $\mathfrak{X}$ admitting vector bundle isomorphisms

$$\varphi_g : \mathcal{E} \xrightarrow{\simeq} \sigma_{\mathfrak{X},g}^*(\mathcal{E}), \ \forall \, g \in G,$$

such that the following diagram commutes for all $g \in G$:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\ \nabla\ } & \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \mathfrak{D}) \\
{\scriptstyle \varphi_g} \downarrow & & \downarrow {\scriptstyle \varphi_g \otimes \eta_g} \\
\sigma^*_{\mathfrak{X},g}(\mathcal{E}) & \xrightarrow{\ \sigma^*_{\mathfrak{X},g}(\nabla)\ } & \sigma^*_{\mathfrak{X},g}(\mathcal{E}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \sigma^*_{\mathfrak{X},g}(\mathfrak{D})),
\end{array}
\tag{4.3.3}
$$

where η_g is the isomorphism defined in (4.3.1).

We shall denote by $\mathbf{Con}^G(\mathfrak{X}, \mathfrak{D})$ the category of G-equivariant logarithmic connections on $(\mathfrak{X}, \mathfrak{D})$.

Remark 4.3.4. Proposition 2.3.11 give rise to isomorphisms of line bundles $\Theta_g : \mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \simeq \sigma^*_{\mathfrak{X},g}\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})$ for each $g \in G$. Now, just as in Lemma 4.2.6, $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})$ admits a canonical logarithmic connection $d(\mathfrak{D})$, which can be constructed by considering étale atlases on $\mathfrak{X}$. It is not difficult to see that this connection is also G-equivariant in the sense of Definition 4.3.2 by taking $\varphi_g = \Theta_g$.

4.4. **Equivariant correspondence.** It is shown in [BL23] that there is an equivalence between the category of logarithmic connections on $(\mathfrak{X}, \mathfrak{D})$ and the category of parabolic connections on (X, D) having weights in $\frac{1}{r}\mathbb{Z}$, and under this equivalence, the category of holomorphic connections on $\mathfrak{X}$ is equivalent to the category of strongly parabolic connections on (X, D). It is given as follows.

(i) [BL23, Definition 4.18] To each logarithmic connection $(\mathcal{E}, \nabla)$ on $(\mathfrak{X}, \mathfrak{D})$, one associates a parabolic connection $(E_\bullet, \widehat{\nabla}_\bullet)$ on (X, D), where

   (a) $E_\bullet$ is the parabolic bundle given as in Theorem 3.1.3, and

   (b) $\widehat{\nabla}_\bullet$ is given by
   $$
   \widehat{\nabla}_{\frac{\ell}{r}} = \pi_*(\nabla(-\ell\mathfrak{D})),
   $$
   where $\nabla(-\ell\mathfrak{D})$ is the tensor product of the connections ∇ an $d(-\ell\mathfrak{D})$.

   (Note that, the push-forward of connection makes sense because the morphism of log-pairs $(\pi, r) : (\mathfrak{X}, \mathfrak{D}) \longrightarrow (X, D)$ is log-étale, see [BL23, Remark 3.8 and Remark 4.19]).

(ii) [BL23, Definition 4.29] On the other hand, for any such parabolic connection $(E_\bullet, \widehat{\nabla}_\bullet)$ on (X, D), one associates the logarithmic connection $(\mathcal{E}, \nabla)$ on $(\mathfrak{X}, \mathfrak{D})$, where
   $$
   \mathcal{E} = \int^{\frac{1}{r}\mathbb{Z}} \pi^*(E_\bullet) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}(\bullet r\mathfrak{D})
   $$
   stands for the coend as in Theorem 3.1.3, and ∇ is the unique connection on the coend $\mathcal{E}$ compatible with the given connections at each term of the coend (the existence of such a connection is guaranteed by [BL23, Lemma 4.27]).

Now we have the following lemma that will be use in the proof of the next theorem.

Lemma 4.4.1. *The induced morphism $\pi^*\Omega^1_{X/\mathbb{C}}(\log(D)) \to \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log(\mathfrak{D}))$ under the log-étale morphism π is an isomorphism.*

Proof. This is basically the stacky version of [BL23, Lemma 3.5], and a similar proof works by considering étale atlases for $\mathfrak{X}$.   □

Theorem 4.4.2. *Let X be an irreducible smooth complex projective variety with an action of a connected affine algebraic group G. Let D be a smooth irreducible Cartier divisor invariant under the G-action. We give $\mathfrak{X}$ the induced G-action from Proposition 2.2.2. Then there is an equivalence between the category of G-equivariant logarithmic connections on $(\mathfrak{X}, \mathfrak{D})$ and the category of G-equivariant parabolic connections on (X, D).*

Proof. Let $\sigma : G \times X \to X$ and $\sigma_{\mathfrak{X}} : G \times \mathfrak{X} \to \mathfrak{X}$ be the action maps. Let $(\mathcal{E}, \nabla)$ be a G-equivariant logarithmic connection on $(\mathfrak{X}, \mathfrak{D})$. This means that we have the following commutative diagram for each $g \in G$ (see Definition 4.3.2):

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\ \nabla\ } & \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \mathfrak{D}) \\
{\scriptstyle \varphi_g}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \varphi_g \otimes \eta_g} \\
\sigma^*_{\mathfrak{X},g}(\mathcal{E}) & \xrightarrow{\ \sigma^*_{\mathfrak{X},g}(\nabla)\ } & \sigma^*_{\mathfrak{X},g}(\mathcal{E}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \sigma^*_{\mathfrak{X},g}(\mathfrak{D}))
\end{array}
\tag{4.4.3}
$$

For each $g \in G$, let

$$
\Theta_g : \mathcal{O}_{\mathfrak{X}}(\mathfrak{D}) \xrightarrow{\ \simeq\ } \sigma^*_{\mathfrak{X},g}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}))
$$

be the isomorphisms resulting from the linearization on $\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})$ in Proposition 2.3.11. The canonical logarithmic connection $(\mathcal{O}_{\mathfrak{X}}(\mathfrak{D}), d(\mathfrak{D}))$ is G-equivariant. For each integer ℓ, we can take the tensor product of ∇ with the canonical connection $d(-\ell\mathfrak{D})$ on $\mathcal{O}_{\mathfrak{X}}(-\ell\mathfrak{D})$, which leads us to the following commutative diagram for all $g \in G$:

$$
\begin{array}{ccc}
\mathcal{E}(-\ell\mathfrak{D}) & \xrightarrow{\ \nabla(-\ell\mathfrak{D})\ } & \mathcal{E}(-\ell\mathfrak{D}) \underset{\mathcal{O}_{\mathfrak{X}}}{\bigotimes} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \mathfrak{D}) \\
{\scriptstyle \varphi_g \otimes \Theta_g^{-\ell}}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle (\varphi_g \otimes \Theta_g^{-\ell}) \otimes \eta_g} \\
\sigma^*_{\mathfrak{X},g}(\mathcal{E}(-\ell\mathfrak{D})) & \xrightarrow{\ \sigma^*_{\mathfrak{X},g}(\nabla(-\ell\mathfrak{D}))\ } & \sigma^*_{\mathfrak{X},g}(\mathcal{E}(-\ell\mathfrak{D})) \underset{\mathcal{O}_{\mathfrak{X}}}{\bigotimes} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \sigma^*_{\mathfrak{X},g}(\mathfrak{D}))
\end{array}
\tag{4.4.4}
$$

Now, recall the proof of Theorem 3.2.4, where we have seen the isomorphism of functors

$$
\sigma^* \circ \pi_* \simeq (\mathrm{Id} \times \pi)_* \circ \sigma^*_{\mathfrak{X}}.
$$

This gives rise to isomorphisms $\sigma^*_g \circ \pi_* \simeq \pi_* \circ \sigma^*_{\mathfrak{X},g}$ for all $g \in G$. Also, by Lemma 4.4.1 we have the isomorphisms:

$$
\pi^* \Omega^1_{X/\mathbb{C}}(\log D) \xrightarrow{\ \simeq\ } \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \mathfrak{D}),
$$
$$
\text{and} \quad \pi^* \Omega^1_{X/\mathbb{C}}(\log \sigma^*_g(D)) \xrightarrow{\ \simeq\ } \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \sigma^*_{\mathfrak{X},g}(\mathfrak{D})).
$$

Applying π_* on the diagram (4.4.4) and using the projection formula, we have the following commutative diagram for each $g \in G$:

$$
\begin{array}{ccc}
E_{\frac{\ell}{r}} & \xrightarrow{\widehat{\nabla}_{\frac{\ell}{r}}} & E_{\frac{\ell}{r}} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}(\log D) \\
\phi_{g,\frac{\ell}{r}} \downarrow \simeq & & \downarrow (\phi_{g,\frac{\ell}{r}}) \otimes \mathrm{Id} \\
\sigma_g^*(E_{\frac{\ell}{r}}) & \xrightarrow{\sigma_g^*(\widehat{\nabla}_{\frac{\ell}{r}})} & \sigma_g^*(E_{\frac{\ell}{r}}) \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}(\log D)
\end{array}
\tag{4.4.5}
$$

This produces a G-equivariant connection $(\widehat{\nabla}_\bullet)$ on the resulting G-equivariant parabolic bundle $E_\bullet$.

Conversely, assume that $(E_\bullet, \widehat{\nabla}_\bullet)$ is a G-equivariant parabolic connection on (X, D). We have already seen in Theorem 3.2.4 that the coend $\mathcal{E} = \int^{\frac{1}{r}\mathbb{Z}} \pi^*(E_\bullet) \otimes \mathcal{O}_{\mathfrak{X}}(\bullet\, r\mathfrak{D})$ admits a G-equivariance structure, and moreover, by [BL23, Lemma 4.27] it admits a unique logarithmic connection ∇ compatible with the given logarithmic connections $\pi^*(\widehat{\nabla}_\bullet) \otimes d(\bullet r\mathfrak{D})$ on the bundles $\pi^*(E_\bullet) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\bullet r\mathfrak{D})$. In fact, it follows from the proof of [BL23, Lemma 4.27] that $(\mathcal{E}, \nabla)$ is the coend of the bifunctor

$$
\left(\pi^*(E_\bullet) \otimes \mathcal{O}_{\mathfrak{X}}(\bullet r\mathfrak{D}), \pi^*(\widehat{\nabla}_\bullet) \otimes d(\bullet\, r\mathfrak{D}) \right) : \left(\frac{1}{r}\mathbb{Z} \right)^{op} \times \left(\frac{1}{r}\mathbb{Z} \right) \longrightarrow \mathrm{Con}(\mathfrak{X}, \mathfrak{D}).
$$

Similarly, for each $g \in G$ we can interpret the connection $\left(\sigma_{\mathfrak{X},g}^*(\mathcal{E}), \sigma_{\mathfrak{X},g}^*(\nabla) \right)$ as a coend of the bifunctor

$$
\left(\sigma_{\mathfrak{X},g}^*(\pi^*(E_\bullet) \otimes \mathcal{O}_{\mathfrak{X}}(\bullet\, r\mathfrak{D})), \sigma_{\mathfrak{X},g}^*(\pi^*(\widehat{\nabla}_\bullet) \otimes d(\bullet\, r\mathfrak{D})) \right) : \left(\frac{1}{r}\mathbb{Z} \right)^{op} \times \left(\frac{1}{r}\mathbb{Z} \right) \longrightarrow \mathrm{Con}(\mathfrak{X}, \mathfrak{D}).
$$

Since $(E_\bullet, \widehat{\nabla}_\bullet)$ is G–equivariant, we have commutative diagrams (4.2.5) for each $h \in G$. Applying π^* to these diagrams, and using $\pi^* \circ \sigma_h^* = \sigma_{\mathfrak{X},h}^* \circ \pi^*$ together with the Lemma 4.4.1, we get a commutative diagram of the form

$$
\begin{array}{ccc}
\pi^*(E_\bullet) & \xrightarrow{\pi^*(\widehat{\nabla}_\bullet)} & \pi^*(E_\bullet) \underset{\mathcal{O}_{\mathfrak{X}}}{\otimes} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \mathfrak{D}) \\
\pi^*(\phi_{h,\bullet}) \downarrow & & \downarrow \pi^*(\phi_{h,\bullet}) \otimes \eta_h \\
\sigma_{\mathfrak{X},h}^*(E_\bullet) & \xrightarrow{\pi^*(\sigma_h^*(\widehat{\nabla}_\bullet))} & \sigma_{\mathfrak{X},h}^*(\pi^*(E_\bullet)) \underset{\mathcal{O}_{\mathfrak{X}}}{\otimes} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \sigma_{\mathfrak{X},h}^*(\mathfrak{D}))
\end{array}
\tag{4.4.6}
$$

where η_h is as in (4.3.1). We now consider the tensor product of the two horizontal arrows in the diagram (4.4.6) with the functor

$$
(\mathcal{O}_{\mathfrak{X}}(\bullet\, r\mathfrak{D}), d(\bullet\, r\mathfrak{D})) : \left(\frac{1}{r}\mathbb{Z} \right)^{op} \longrightarrow \mathrm{Con}^G(\mathfrak{X}, \mathfrak{D}),
$$

which will give rise to diagrams of the form

$$
\begin{array}{ccc}
\pi^*(E_\bullet) \underset{\mathcal{O}_{\mathfrak{X}}}{\otimes} \mathcal{O}_{\mathfrak{X}}(\bullet r\mathfrak{D}) & \xrightarrow{\pi^*(\widehat{\nabla}_\bullet)\otimes d(\bullet\, r\mathfrak{D})} & \pi^*(E_\bullet) \underset{\mathcal{O}_{\mathfrak{X}}}{\otimes} \mathcal{O}_{\mathfrak{X}}(\bullet r\mathfrak{D}) \underset{\mathcal{O}_{\mathfrak{X}}}{\otimes} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \mathfrak{D}) \\
\Big\downarrow{\scriptstyle \pi^*(\phi_{h,\bullet})\otimes\Theta_h^{\otimes\bullet}} & & \Big\downarrow{\scriptstyle (\pi^*(\phi_{h,\bullet})\otimes\Theta_h^{\otimes\bullet})\otimes\eta_h} \\
\sigma^*_{\mathfrak{X},h}(E_\bullet) \underset{\mathcal{O}_{\mathfrak{X}}}{\otimes} \sigma^*_{\mathfrak{X},h}(\mathcal{O}_{\mathfrak{X}}(\bullet\, r\mathfrak{D})) & \xrightarrow{\pi^*(\sigma^*_h(\widehat{\nabla}_\bullet))\otimes d(\bullet r\mathfrak{D})} & \sigma^*_{\mathfrak{X},h}(\pi^*(E_\bullet)) \underset{\mathcal{O}_{\mathfrak{X}}}{\otimes} \mathcal{O}_{\mathfrak{X}}(\bullet\, r\mathfrak{D}) \underset{\mathcal{O}_{\mathfrak{X}}}{\otimes} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \sigma^*_{\mathfrak{X},h}(\mathfrak{D}))
\end{array}
$$

where Θ_h is as in (4.3.4). Each term in the diagram above is a bifunctor

$$
\left(\frac{1}{r}\mathbb{Z}\right)^{\mathrm{op}} \times \left(\frac{1}{r}\mathbb{Z}\right) \longrightarrow \mathbf{Con}(\mathfrak{X},\mathfrak{D}).
$$

Finally, taking the coends of these bifunctors, we get a commutative diagram of the form

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\quad\nabla\quad} & \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \mathfrak{D}) \\
\Big\downarrow{\scriptstyle \varphi_h} & & \Big\downarrow{\scriptstyle \varphi_h\otimes\eta_h} \\
\sigma^*_{\mathfrak{X},h}(\mathcal{E}) & \xrightarrow{\sigma^*_{\mathfrak{X},h}(\nabla)} & \sigma^*_{\mathfrak{X},h}(\mathcal{E}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathbb{C}}(\log \sigma^*_{\mathfrak{X},h}(\mathfrak{D}))
\end{array}
$$

Thus, under the correspondence of [BL23] (see the beginning of §4.4), the resulting logarithmic connection $(\mathcal{E},\mathfrak{D})$ is G-equivariant. This completes the proof. $\qquad\square$

Remark 4.4.7. For $\lambda \in \mathbb{C}$, there is a notion of *logarithmic λ–connections* on X, which are pairs (E,∇), where E is a vector bundle on X and $\nabla : E \longrightarrow E \otimes \Omega^1_{X/\mathbb{C}}(\log D)$ is a $\mathbb{C}$-linear homomorphism of sheaves satisfying the λ–*twisted Leibniz rule*:

$$
\nabla(fs) = f\nabla(s) + \lambda \cdot df \otimes s,
$$

for all locally defined sections f and s of $\mathcal{O}_X$ and E, respectively. Such connections can be twisted by a Cartier divisor B with support in D. The notion of logarithmic λ–connection gives rise to the notion of *parabolic λ–connection* generalizing the Definition 4.2.2. When X is endowed with an action of a connected affine algebraic group G having no non-trivial characters, and D is invariant under the given G–action on X, we may talk about G–equivariant parabolic λ–connection as well. All these notions have their appropriate counterparts on the associated r-th root stack $\mathfrak{X} = \mathfrak{X}_{(\mathcal{O}_X(D),s_D,r)}$. Then Theorem 4.4.2 can be generalized in the setup of G–equivariant λ–connections following the same line of arguments. The case of $\lambda = 0$ is of particular interest, as it leads to the notion of *Higgs bundles*. One may use this to generalize the Biswas-Borne correspondence for Higgs bundles (see [BMW13]) in G–equivariant setup.

References

[BH13]   Indranil Biswas and Viktoria Heu. On the logarithmic connections over curves. *J. Ramanujan Math. Soc.*, 28A:21–40, 2013. [↑ 18.]

[Bis97]   Indranil Biswas. Parabolic bundles as orbifold bundles. *Duke Math. J.*, 88(2):305–325, 1997. doi:10.1215/S0012-7094-97-08812-8. [↑ 2 and 12.]

[BL23]   Niels Borne and Amine Laaroussi. Parabolic connections and stack of roots. *Bull. Sci. Math.*, 187:Paper No. 103294, 33, 2023. doi:10.1016/j.bulsci.2023.103294. [↑ 2, 3, 18, 19, 20, 21, 22, and 23.]

[BMW12]   Indranil Biswas, Souradeep Majumder, and Michael Lennox Wong. Root stacks, principal bundles and connections. *Bull. Sci. Math.*, 136(4):369–398, 2012. doi:10.1016/j.bulsci.2012.03.006. [↑ 2.]

[BMW13]   Indranil Biswas, Souradeep Majumder, and Michael Lennox Wong. Parabolic Higgs bundles and Γ-Higgs bundles. *J. Aust. Math. Soc.*, 95(3):315–328, 2013. doi:10.1017/S1446788713000335. [↑ 23.]

[Bor07]   Niels Borne. Fibrés paraboliques et champ des racines. *Int. Math. Res. Not. IMRN*, (16):Art. ID rnm049, 38, 2007. doi:10.1093/imrn/rnm049. [↑ 2, 3, 8, 12, 13, 15, and 16.]

[Cad07]   Charles Cadman. Using stacks to impose tangency conditions on curves. *Amer. J. Math.*, 129(2):405–427, 2007. doi:10.1353/ajm.2007.0007. [↑ 3 and 4.]

[CM24]   Sujoy Chakraborty and Souradeep Majumder. Orthogonal and symplectic parabolic connections and stack of roots. *Bull. Sci. Math.*, 191:Paper No. 103397, 27, 2024. doi:10.1016/j.bulsci.2024.103397. [↑ 2.]

[CMW18]   Sebastian Casalaina-Martin and Jonathan Wise. An introduction to moduli stacks, with a view towards Higgs bundles on algebraic curves. In *The geometry, topology and physics of moduli spaces of Higgs bundles*, volume 36 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 199–399. World Sci. Publ., Hackensack, NJ, 2018. [↑ 3 and 7.]

[CP24]   Sujoy Chakraborty and Arjun Paul. Real structures on root stacks and parabolic connections. *Geom. Dedicata*, 218(2):Paper No. 35, 28, 2024. doi:10.1007/s10711-023-00880-1. [↑ 2.]

[Dol03]   Igor Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003. doi:10.1017/CBO9780511615436. [↑ 5.]

[DP22]   Ron Donagi and Tony Pantev. Parabolic Hecke eigensheaves. *Astérisque*, (435):viii+192, 2022. doi:10.24033/ast.1178. [↑ 2.]

[EV92]   Hélène Esnault and Eckart Viehweg. *Lectures on vanishing theorems*, volume 20 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1992. doi:10.1007/978-3-0348-8600-0. [↑ 18 and 19.]

[Har77]   Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. [↑ 5.]

[Hei10]   Jochen Heinloth. Lectures on the moduli stack of vector bundles on a curve. In *Affine flag manifolds and principal bundles*, Trends Math., pages 123–153. Birkhäuser/Springer Basel AG, Basel, 2010. doi:10.1007/978-3-0346-0288-4_4. [↑ 11.]

[LSS13]   Frank Loray, Masa-Hiko Saito, and Carlos Simpson. Foliations on the moduli space of rank two connections on the projective line minus four points. In *Geometric and differential Galois theories*, volume 27 of *Sémin. Congr.*, pages 117–170. Soc. Math. France, Paris, 2013. [↑ 2.]

[MFK94]   D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994. [↑ 5.]

[MS80]   V. B. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Math. Ann.*, 248(3):205–239, 1980. doi:10.1007/BF01420526. [↑ 2.]

[MY92]   M. Maruyama and K. Yokogawa. Moduli of parabolic stable sheaves. *Math. Ann.*, 293(1):77–99, 1992. doi:10.1007/BF01444704. [↑ 2.]

[Ols07]   Martin Olsson. Sheaves on Artin stacks. *J. Reine Angew. Math.*, 603:55–112, 2007. doi:10.1515/CRELLE.2007.012. [↑ 3 and 11.]

[Ols16]   Martin Olsson. *Algebraic spaces and stacks*, volume 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016. doi:10.1090/coll/062. [↑ 10 and 15.]

[Rom05]   Matthieu Romagny. Group actions on stacks and applications. *Michigan Math. J.*, 53(1):209–236, 2005. doi:10.1307/mmj/1114021093. [↑ 6.]

[Sim90]   Carlos T. Simpson. Harmonic bundles on noncompact curves. *J. Amer. Math. Soc.*, 3(3):713–770, 1990. doi:10.2307/1990935. [↑ 2.]

[Sta24]   The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2024. [↑ 11 and 13.]

[Yok95]   Kôji Yokogawa. Infinitesimal deformation of parabolic Higgs sheaves. *Internat. J. Math.*, 6(1):125–148, 1995. doi:10.1142/S0129167X95000092. [↑ 12.]